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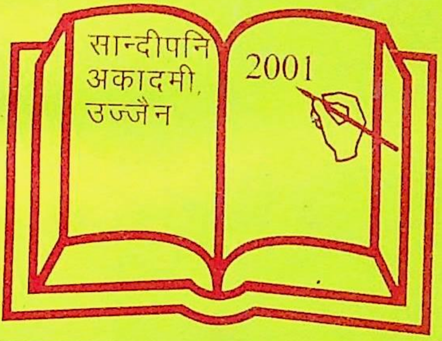
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ON CERTAIN CIPHERING SYSTEMS ON M_n -MATRICES

A. H. Ahmed, R. N. Mohan & R. Bhanu Teja

(Received 26/12/2001)

ABSTRACT. In this paper, by using M_n-Matrices, which are of combinatorial interest, certain cipher systems are obtained.

Keywords. Cipher systems, Cipher-path, combinatorial, algorithm, M_n-Matrix.
C.R. Categories: G.2.1

1. INTRODUCTION

There is the strongest interplay amidst M_n-Matrices, combinatorial designs, codes and graphs. Various kinds of matrices namely Hadamard matrices, conference matrices, circulent matrices, Siedal matrices and so on are some of the important matrices in these lines. Primarily these matrices are very much used to construct codes. Recently Mohan [3] defined a certain family of matrices called M_n-Matrices and also mentioned their use in various combinatorial configurations there in. For example by making use of these M_n-Matrices, Mohan and Ahmed [2] gave a construction for authentication/ secrecy codes.

In this paper we suggest that these M_n-Matrices can also be used to implement certain cipher systems as given by Anderson [1] and Frank [5,6]

2. THE CIPHER DESIGN

The most commonly used poly alphabetic cipher is

$$C_k = P_k + K_k \pmod{26}, \tag{2.1}$$

where P_k is the plain text, K_k is the key stream and C_k is the resulting cipher text. The key stream is just a short key word repeatedly indefinitely.

For the modern versions using computers, characters are represented in 8-bit EBCDIC or ASCII code and the arithmetic is done modulo 256 rather than modulo 26, that is

$$C_k = P_k + K_k \pmod{256} \tag{2.2}$$

A pseudo-random number generator most usually generates the key stream. These ciphers are easily defeated as shown in Anderson [2], and Rubin [4,5] as it is possible to separate the key stream from the message stream. Therefore the plain text and the key must be combined in a way that cannot be reversed. The simplest way to do this is to perform simple substitution on either the plain text or on the cipher text.

That is done by a permutation of the numbers representing the characters as follows·

$$C_k = T\,(S\,(P_k) + K_k) \pmod{256} \tag{2.3}$$

where T(x) and S(x) represent permutations of the number representing the character x. This cipher is known as a Type (iv) quagmire cipher and was invented by hoppyists in American Cryptogram Association.

3. IMPLEMENTATION OF THE CIPHER SYSTEM ON M_n-MATRICES

To implement the above cipher we need to define the permutations T and S on the family of matrices called M_n-Matrices. These matrices have useful combinatorial features, needed for our purposes.

In Mohan [3], M_n-Matrices are defined as $M_n = (1 + (i-1)\,(j - 1))$ mod n. This can be further generalized as $M_n = (i.j)$ mod n. In both the cases we consider the matrices with n as prime only in order to have the occurrence of the elements in the way that we need.

$M_n = \{a_{ij}\}$ where
$a_{ij} = (ij) \bmod n,\ 1 \le i, j \le n-1$ and where n is prime (3.1)

This matrix has an important combinatorial property that each element comes once and only once in each row and each column. Besides every column and every row have distinct elements. Moreover the distribution of the elements within the rows or the columns is not with obvious systematic names. So we can use the matrix by taking n = 257 by making use of the symbols 0,1,2,3...255. However for 8 bits representation, the smallest representation is (0,0,0,0,0,0,0,0) and the largest representation is (1,1,1,1,1,1,1,1,1) then n = 257. So for ASCII or EBCDIC we can take n = 257. As it is desired by deleting 0 we consider the remaining 256 symbols then n = 256. Whatever the case may be we construct our matrix with n as a prime number only, as we need a particular combinatorial property, also with n-1 elements in the plain text P.

Explanation: Construct the M_n-Matrix as mentioned above.

For permutation T

First make a substitution of the symbols called the plain text P: (a, b, c, d...n-1 symbols) with any column say C_i of the matrix having n-1 symbols in each row and each column. Then symbols of that column C_i are substituted with their corresponding encoding rule - the row numbers, which take them to any other column C_j. That is consider any symbol a_{ij} in C_i the chosen column, note the encoding rule e (a_{ij}), the row number in which this symbol occurs in the other chosen column C_j and form the new set say C_k with those encoded symbols e (a_{ij}) s. i.e., Row numbers of aij s of C_i in the C_j that leads to the set C_k. Then we apply the key as follows:

$$P_m = (C_k + K_k) \bmod n-1 \qquad (3.2)$$

For permutation S

Now apply the encoding rule over the symbols of this P_m, to take them to the column C_t and enlist them to form the final cipher system required i.e. R.

For deciphering

We follow the same path in the reverse order. We use the inverse functions and reverse key to get our original plain text P from the final ciphered system R.

The cipher-path used in this example: For any text P we permute the ASCII or EBCDIC table using a column of M_n-Matrix say C_i th column to get the partial ciphered text C_k . Then we apply the key over C_k to get a new table P_m. Again we permute the elements in the new table P_m by using a column of M_n-Matrix, say C_t to get the final cipher system R. For deciphering traces it back to get our original plain text P from the final cipher system R.

In section 5 we will discuss a generalized Cipher-Path.

4. ILLUSTRATION

For a simple illustration consider n = 5, a prime number. The M_n-Matrix is given by

$$M_n = (i.j) \bmod n\text{-}1,\ 1 \le i,j \le n\text{-}1$$

which gives

$$M_n = (\ i.j\ ) \bmod 4,\ 1 \le i,j \le 4$$

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{matrix}$$

Now consider a code table of 4 symbols say

A
B
C
D

Comparing with the number representation of the first column C_1 of M_n-Matrix, given above.

1
2
3
4

Now for applying the encoding rule e(x), we start the permutation: $[T(C_1) = C_3]$. i.e., on the elements of C_1 with reference to the row numbers of these elements in any other column say C_2. That is the symbol in the column 1 in C_1 is in the third position in C_2. So e (1) = 3, similarly e (2) = 1, e (3) =4, e (4) = 2. So we get, C_3 as

3
1
4
2

Now on this C_3 we apply the key K_k following the principle that $P_m = (C_k + K_k) \bmod n\text{-}1$. Let the required key in the present case be some 6. So $P_m = (C_3 + 6) \bmod 4$, gives out

1
3
2
4

On the elements of this P_m we apply the permutation S, with reference to the row numbers of the column 3, say C_4. S(1) = 2, S(3) = 1, S(2) = 4, S(4) = 3. Thus $S(P_m) = R$, given by

2
1
4
3

This R is final cipher system.

For deciphering

Consider the inverse permutations T^{-1} and S^{-1} and the reverse key $(K_k)^{-1}$. All these are to be applied in the reverse order to obtain the plain text P from the cipher system R.

First consider R and the inverse of the permutation of S over the elements of R, i.e. , $S^{-1}(R)$, we get $S^{-1}(2) = 1$, $S^{-1}(1) = 3$, $S^{-1}(4) = 2$, $S^{-1}(3) = 4$. From this we obtain P_m. Now we get C_3 with the formula $C_k = (P_m - K^{-1}_k) \bmod n\text{-}1$, i.e., $C_3 = (P_m - 6) \bmod 4$. Now $T^{-1}(C_3) = C_1$ leads to the present plain text P.

This algorithm can easily be computerized. What is that needed is to fix up the number n as prime, and constructing the M_n-Matrix, fixing up the column number in it from which the permutation is to be started. Then the permutation goes in sequence using two successive columns with every text. This we have done in the coming discussions in this paper itself.

5. A GENERALIZED CIPHER PATH

In this paper we aim at establishing the prime role of these M_n-Matrices, in cipher systems. The functions defined here can be further complicated, if more secrecy is needed. When high degree of secrecy is demanded, more complexities can be infused in the ciphering and also can be computerized. The well-defined cipher-path of an M_n-Matrix can be utilized for the compartmentalized and Hierarchical secret sharing schemes for instance, see Mohan and Ahmed [4].

The generalized cipher-path can be defined as follows:

$$n\text{-}a_1 - C_1 - O_1 - k_1 - O_2 - C_2 - O_3 - k_2 - O_4 - C_3 - O_5 - \ldots C_n - O_n - k_n - S$$

$$n - C_1 - RC_3 - CT_1 - k - CT_2 - RC_4 - FCT$$

$$n - 1 - 3 - O_1 - 6 - O_2 - 4 - S$$

$$5 - 1 - 3 - O_1 - 6 - O_2 - 4 - S \qquad (5.1)$$

Explanation-n is any prime number, to be chosen to construct an Mn-Matrix (i.j) mod n, with $1 \le i, j \le n - 1$. Consider a plain text and compare to C_1 the first column - cipher that with the third column (writing the corresponding row numbers in the third column for the elements of the first column), and this is the first cipher text, and apply the key k and this is the second cipher text, for this again cipher with the fourth column. This resultant text is the final cipher text required.

Notation 3, 4 (bold) indicate - writing row numbers in that column, for the elements in the given column. Small font of k indicates key. O_1, O_2 indicate first cipher text and the second cipher text respectively. S indicates final cipher text.

Exercise: Construct the cipher system by using M_n-Matrix with the given cipher-path with addition as a binary operation.

$$7 - 2 - 5 - O_1 - 3 - 4 - O_2 - 3 - 5 - 1 - S \tag{5.2}$$

Now one can construct a more complicated cipher system best suited for one's complex situations to maintain high order of secrecy. And that the well-defined path can be used for a compartmentalized and Hierarchical secret sharing schemes also as explained in
Mohan and Ahmed [4]. For example consider the path given in 5.1 above and allot serial numbers and they are denoted as

1.5, 2.1, 3.3, 4.O_1, 5. 6 , 6.O_2 , 7. 4, 8. S

which can be used for the above purpose. As this cipher system is so flexible to one's requirements, it can be modified and can be made more critical to one's need, protecting the secret text or cipher text from the attacks of the enemy or from any contamination.
If k is any key, it can be applied on the elements with any binary operation $*$ mod n to be defined.

In a cipher path P. the number of times the cipher used denotes the order of the path denoted by 'm', of the path and the number of times the key/s used is called the degree K of the cipher path. In the path (5.1) the order m = 2 and the degree K = 1 and in the path (5.2) the order m = 4 and the degree K = 2 and the binary operation in both the cases is '+' an ordinary addition mod n.

Then construct a cipher-path P of order 6 and degree 4 over the binary operation '+' on the M_n-Matrix n = 11 and thus find the final cipher text with the defined path. Construct P7(6,4) over the binary operation 'X' the ordinary multiplication. Finally a cipher-path of order m and degree K over the defined binary operation '$*$', on the M_n-Matrix n is denoted by Pn(m,K) over the binary operation '$*$'. Any number of binary operations can be used in the same cipher-path. In the order and degree of a cipher-path, the following points are to be noted.

In the M_n-Matrix taken over n as a prime we can consider M_n = (i.j) mod n-1 and hence there are n-1 columns only. We choose one column as a plain text so there remain n-2 columns only, and we cipher the chosen plain text in n-2 ways. So the limits of the order m are in the range $1 \leq m \leq n\text{-}2$. The repetition of columns for ciphering is prohibited to avoid confusion. However infinite number of keys can be used with mod n condition. Hence the range of K is $1 \leq K \leq \propto$.

Here we give a computer programming in C language for the ciphering of an M_n-Matrix for a given n, m, K over the defined binary operation say '+' ordinary addition mod n.

The superiority of these M_n-Matrices over Latin squares is owing to their more generalized nature than that of Latin squares. However both of these combinatorial configurations are the best suited for the construction of cipher systems.

By bringing a sequence of permutations to play their role along with the family of these M_n-Matrices, one can construct more complex cipher systems.

6. FLOW CHART

Computer program in C language for the cipher system.

I.

1. Give the prime number
2. Construct the M_n-Matrix (i, j) mod n, $1 \leq i, j \leq n-1$.
3. Construct the plain text as a column with n-1 rows
4. Compare with any column a_1
5. Cipher with any other column C_1
6. Write the resultant text
7. Apply key k over $*$ given
8. Write the resultant text
9. Repeat the process for the given number of times, maximum for n-2
10. Print the result
11. End

II.

1. Given the prime number n
2. Mention what $*$ means
3. If more operations are needed mention them
4. Mention the number of times the process to be repeated.

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NEW CLASSES OF FINITE GROUPOIDS USING Z_n

W.B. Vasantha Kandasamy

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ABSTRACT. In this paper we introduce four new classes of groupoids using Z_n, n is a finite positive integer. We obtain a necessary and sufficient condition for the groupoids in these classes to be idempotent groupoids. Secondly, we find conditions for these groupoids to be semigroups. Since all semigroups are groupoids; the study of groupoids which are the most generalized concept of semigroups is carried out as such it is not found in literature. Here we define several new type of groupoids as in case of loops.

Keywords. Groupoids, Bol groupoid, P-groupoid, Moufang groupoid, Idempotent groupoid and Alternative groupoid.

1. INTRODUCTION

The study of groupoid is very meagre, the main reason that can be attributed to this is that so far we don't see in literature groupoids which are naturally constructed. Further Groupoids are nothing but semi groups, which are non-associative. Hence the study of these generalized classes is not only interesting but will also prove to be useful. The paper consist of five sections. Section 1 is introductory in nature. In section 2 we define some new concepts and identities in groupoid. In section 3 we introduce the new classes of groupoids using Z_n. Some interesting results and theorems are proved in section 4. Section 5 suggests some open problems.

Definition (Bruck) Given an arbitrary non empty set P a mapping of P × P into P is called a binary operation on P. Given such a mapping $\sigma : P \times P \rightarrow P$ we use it to define a product $*$ in P by declaring $a * b = c$ if $\sigma(a,b) = c$.

Definition (Bruck) A non empty set of elements G is said to form a groupoid if in G is defined a binary operation called the product denoted by $*$ such that $a * b \in G$ for all $a,b \in G$.

A groupoid is said to be commutative if for all $a,b \in G$: $a*b = b*a$. A groupoid is said to have an identity element e if $a*e = e*a = a$ for all $a \in G$ ($e \in G$). We call the order of the groupoid G to be the number of distinct elements in G and denote it by o(G). If G has only finite number of elements then we say G is a finite groupoid, otherwise G is said to be an infinite groupoid. (For details see [1]).

2. SOME NEW CONCEPTS AND IDENTITIES IN A GROUPOID

In this section, we define new concepts in any groupoid G like right ideal, left ideal, ideal, normal sub groupoid, simple groupoid, direct product of Groupoids and normal groupoid. Further, as in the case of loops we extend the identities in loops to Groupoids. Groupoids are the generalized algebraic structures of loops as every loop is a groupoid and a groupoid in general is not a loop.

Definition 2.1 Let (G,$*$) be a groupoid. P a proper non empty subset of G. P is said to be sub groupoid if (P,$*$) is itself a groupoid.

Definition 2.2 Let (G.$*$) be groupoid. A non empty subset P of G is called a left ideal of the groupoid G if

1. P is a sub groupoid
2. For $x \in G$ and $a \in P$, $xa \in P$.

We can similarly define right ideal of G. P is called an ideal if P is simultaneously a left and a right ideal of G.

Definition 2.3 Let G be a groupoid. A sub groupoid V of G is said to be a normal sub groupoid if

1. $aV = Va$
2. $(Vx)y = V(xy)$
3. $y(xV) = (yx)V$ for all x,y,a in G.

Definition 2.4 A groupoid G is said to be simple if it does not contain any non trivial normal sub groupoids.

Definition 2.5 Let (G,∗) and (G1,) two groupoid. The direct product of GxG1 is a groupoid defined as follows:

Definition 2.6 Let G be a groupoid we say G is a normal groupoid if

1. xG = Gx
2. G(xy) = (Gx)y
3. y(xG) = (yx)G, for all x,y ∈ G.

Now we proceed on to define groupoid, which satisfy the special identities.

Definition 2.7 A groupoid G is said to be a Moufang groupoid if the identity (x,y)(z,x)=(x(yz))x is true for all x,y,z in G.

Definition 2.8 A groupoid G is said to be a Bol groupoid if the identity ((xy)z)yx((yz)y) is true for all x,y,z in G.

Definition 2.9 A groupoid G is said to be right alternative if (xy)y = x(yy) for all x,y ∈ G. Similarly we can define left alternative groupoid (xx)y = x(xy) a groupoid G is alternative if it is both right and left alternative.

Definition 2.10 Let G be a groupoid. G is said to be strictly non commutative if xy ≠ yx for any x, y ∈ G, x≠ y

Definition 2.11 Let G be a groupoid. G is said to be a P-groupoid if (xy)x = x(yx) for all x, y∈ G.

In the next section, we introduce the new classes of groupoids using Z_n.

3. NEW CLASSES OF GROUPOIDS USING Z_n

In this section we define the new classes of groupoid constructed using Z_n and give some examples. Throughout this paper $Z_n = \{0,..,n-1\}$ where $n < \infty$ and $n \geq 3$.

Definition 3.1 Let $Z_n = \{0,1,2,.....n-1\}$. Take $t,u \in Z_n/\{0\}$ such that $t \neq u$. $(t,u)=1$. Define a binary operation $*$ on Z_n by $a*b = at + bu$ (mod) for all $a,b \in Z_n$ $\{Z_n,*,(t,u)\}$ is a groupoid which we denote by $Z_n(t,u)$.
Now as we vary the pair $t,u \in Z_n\backslash\{0\}$ we get a collection of groupoid for a fixed n. We denote this collection of groupoids by $Z(n)$.

Example 3.1 Consider $Z_6 = \{0,1,2,3,4,5\}$, Let $t = 3$ and $u = 4$. Now define $*$ on Z_6 by $a*b = 3a + 4b$ (mod n)$\{Z_6,*,(3,4)\}$ is a groupoid.
Now when we take the pair (4,3) instead of (3,4) we get another groupoid different from $\{Z_6,*, (3,4)\}$, Thus in general $Z_n(t,u)$ is not the same $Z_n(u,t)$.

Remark Now using Z_6 we construct 18 groupoids. So the class $Z(6)$ contains 18 groupoids constructed using definition 3.1.

Example 3.2 Let $Z_7 = \{0,1....6\}$, $Z_7(5,2)$ is groupoid, which is not a semigroup. Now we define a little generalized class of groupoid which contains the class of groupoid $Z(n)$ as follows.

Definition 3.2 Let $Z_n = \{0,1....n-1)$ For $t,u \in Z_n/\{0\}$ $t \neq u$. Define a binary operation $*$ on Zn by $a*b = ta + ub$(mod n) $\{Z_n,*,(t,u)$ is a groupoid.
Now for varying $t,u \in Z_n \backslash\{0\}$, $t \neq u$ we get a collection of groupoid for a fixed n which we denoted by $Z*(n)$. Clearly $Z(n) \subset Z*(n)$.

Example 3.4 Let $Z_8 = \{0,1,2......7\}$. Define for $3,6 \in Z_8$ the operation on the groupoid Z_8 as $a*b = 3a + 6b$ (mod n) for all $a,b \in Z_8$. $\{Z_8,*,(3,6)\}$ is groupoid. Note this groupoid $\{Z_8,*,(3,6)\}$ does not belong to $Z(8)$.

Definition 3.3 Let $Z_n = \{1,2.....n-1\}$. Take $t,u \in Z_n/\{0\}$. Define a binary operation $*$ on Z_n by $a*b = ta + ub$ (mod n). $\{Z_n,*,(t,u)\}$ is a groupoid.
By varying (t,u) we get a collection of groupoids for a fixed n. We denote this collection by $Z^{**}(n)$. Note the class of groupoids $Z^{*}(n) \subset Z^{**}(n)$. Here we can have $t = u$ also. Thus we have $Z(n) \subset Z^{*}(n) \subset Z^{**}(n)$.

Example 3.5 Let $Z_{12} = \{0,1,2,3,4.....11\}$, Take $t = u = 6$. We define for all a,b in Z_{12} the operation $*$ by $a*b = 6a + 6b \pmod{12}$. Clearly $\{Z_{12}, *, (6,6)\}$ is a groupoid.

Now we define yet another class of groupoids which is the most generalized one.

Definition 3.4 Let $Z_n = \{0,1,\ldots,n-1)$, Take $t,u \in Z_n$ define $*$ on Z_n by $a*b = ta + ub \pmod{n}$ for all $a,b \in Z_n$. $\{Z_n, *, (t,u)\}$ is a groupoid. Clearly in this definition one of t or u can also take the value zero.

Example 3.6 Let $Z_{11} = \{0,1,\ldots\ldots 10\}$, Take u = 0 and t = 3 define $*$ on Z_{11} by $a*b = 3a \pmod{n}$. Clearly $\{Z_n, *, (3,0)\}$ is a groupoid.
We denote the class of groupoid got by varying t,u and for a fixed n by $Z^{***}(n)$. We notice that $Z^{**}(n) \subset Z^{***}(n)$. This is the most generalized class and we have for any Z_n, n a positive integer $Z(n) \subset Z^{*}(n) \subset Z^{**}(n) \subset Z^{***}(n)$.

In the following section, we study some of the properties of these classes of groupoids.

4. SOME PROPERTIES OF THE NEW CLASSES OF GROUPOIDS

Here we obtain some interesting results and necessary and sufficient conditions for a groupoid to be an idempotent groupoid.

Example 4.1 Let $Z_4 = \{0,1,2,3\}$, the groupoid $Z_4(2,3)$ has (0,2) to be a left ideal which is not a right ideal.

Theorem 4.1 Let $Z_n(t,u)$ be a groupoid of order n. P is a left ideal of $Z_n(t,u)$ if and only if P is right ideal of $Z_n(u,t)$.

Proof. The result follows easily from the definition of groupoids $Z_n(t,u)$, $Z_n(u,t)$ and the left and right ideals of a groupoid.

Example 4.2 Consider the groupoid Z_6 (2,4). It can be checked that $P = \{0,2,4\}$ is an ideal of $Z_6(2,4)$. Also $P = \{0,2,4\}$ is normal sub groupoid of $Z_6(2,4)$.

Theorem 4.2 Let $Z_n(t,u) \in Z((n)$. Now $a*a = 0 (\text{mod } n)$ for all $a \in Z_n$ is and only if $t + u = 0(\text{mod } n)$.

Proof. Given $Z_n(t,u) \in Z(n)$ implies $t \neq u$, $t,u \in Z_n/\{0\}$ and $(t,u)=1$. Let $a \in Z_n$ $a*a = ta + ua(\text{mod } n)$. Now $a*a = ta + ua (\text{mod } n) = 0(\text{mod } n)$ that is $(t + u)a = 0(\text{mod } n)$ for all $a \in Z_n$ this is true if and only if $t + u = 0(\text{mod } n)$.

Remark If $Z_n(t,u)$ is in $Z^*(n)$ of $Z^{**}(n)$ then the theorem holds good. If $Z_n(t,u)$ belong to $Z^{***}(n)$ then clearly both t and u must be non zero elements of Z_n otherwise $a*a=0(\text{mod } n)$ cannot occur.

Theorem 4.3 Let $Z_n(t,u) \in Z(n)$. Now $a*a = a(\text{mod } n)$ for all $a \in Z_n$ if and only if $(t + u - 1) = \text{mod } n)$.

Proof. Given $Z_n(t,u) \in Z(n)$. Now $a*a = ta + ua(\text{mod } n) = (t + u)a(\text{mod } n)a*a = a(\text{mod } n)$ if and only if $(t + u)a = a(\text{mod } n)$ for all a. This is possible if and only if $t + u - 1 = 0(\text{mod } n)$ Hence the claim.

Remark. If $Z_n(t,u) \in Z^*(n)$ we get the same result. But if $Z_n(t,u) \in Z^{**}(n)$ and if $t = u$ then we see that $(2u-1) = 0(\text{mod } n)$. that is $2u = 1(\text{mod } n)$ this is possible only when n is odd. When n is even this will never occur. So in case $Z_n(t,u) \in Z^{**}(n)$ and if $t = u$ we have $a*a = a(\text{mod } n)$ can occur only when n is odd. Clearly whatever be n if one of t or u is zero we cannot have $a*a = a(\text{mod } n)$.

Theorem 4.4 The groupoid $Z_n(0,t)$ is never a normal groupoid of $Z^{***}(n)$ for $t \in Z_n/\{0,1\}$.

Proof. Clearly $x\, Z_n(0,t) \neq Z_n(0,t)\, x$ for all $x \in Z_n(0,t)$. Hence the claim.

Example 4.3. Let $Z_6(0,2)$ be the groupoid with operation $a*a = 0a + 2b(\text{mod } 6)$. Now $2.Z_6(0,2)=(0,2,4)$ but $Z_6(0,2).2 = 4$. Hence $Z_6(0,2)$ is not a normal groupoid.

Theorem 4.5 The groupoids $Z_n(t,t)$ for $t \in Z_n/\{0\}$ are normal groupoids of $Z^{**}(n)$.

Proof. It is easily verified using definition of the normal groupoid.

Theorem 4.6. The groupoid $Z_n(t,t)$ for $t \in Z_n/\{0\}$ are commutative groupoids.

Proof. Clearly from the very definition of $Z_n(t,t)$.

Corollary 4.7 The groupoids $Z_n(0,t)$ are non commutative groupoids of $Z^{***}(n)$ for all $t \in Z_n/\{0\}$.

Proof. Easily verified as $a*b = 0a + tb \pmod{n}$ and $b*a = 0b + ta \pmod{n}$ if $a \neq b$ we have $a*b \neq b*a$. hence the claim.

Theorem 4.8 The groupoids $Z_n(t,t)$ are P-groupoids.

Proof. A groupoid is said to be a P-groupoid if $(x*y)*x = x*(y*x)$ for all $x,y \in Z_n(t,t)$.
Now $(x*y)\,x = (tx + ty)x = t^2x + t^2y + tx \pmod{n}$.
$X*(y*x) = x(*(ty + tx) = tx + t^2y + t^2x \pmod{n}$. Thus $Z_n(t,t)$ is a P-groupoid.

Theorem 4.9 The groupoid $Z_n(t,t)$ in $Z(n)$ is a P-groupoid when $t^2 \equiv t \pmod{n}$ and $u^2 \equiv u \pmod{n}$ and n a non prime.

Proof. $Z_n(t,u)$ is a P-groupoid only if $(x*y)*x = x*(y*x)$ for all $x,y \in Z_n$.
$(x*y)*x = (tx + uy)*x = t^2x + tuy + ux$,
$x*(y*x) = x*(ty + ux) = tx + uty + u^2x$
If $(x*y)*x = x*(y*x)$ for all $x,y \in X_n$ we must have $t^2x + tuy + ux + uty + u^2x \pmod{n}$ that is $(t^2 - t)x \equiv 0 \pmod{n}$.

Thus if $t^2 \equiv t \pmod{n}$ and $u^2 \equiv u \pmod{n}$ we have $Z_n(t,u)$ is a P-groupoid. The condition n is non prime is essential for n is a prime we cannot have $t^2 \equiv t \pmod{n}$ and $u^2 \equiv u \pmod{n}$.

Example 4.4 Let $Z_5(2,3) \in Z(5)$; $Z_5(2,3)$ is not a P-groupoid for $(1*2)*1 \neq 1*(2*1)$.

Theorem 4.10 The groupoid $Z_n(0,t)$ in which $t^2 \equiv t \pmod{n}$ are Bol.Moufang and alternative groupoids.

Proof. Given $t^2 \equiv t \pmod n$ and $t \in Z_n$. $Z_n(0,t)$ is a Bol groupoid if $((x*y)*z)*y = x*((y*z)*y$ for all x,y,z in $Z_n(0,t)$
$((x*y)*z)*y = (ty*z)*y = tz*y \pmod n = ty \pmod n$
$x*((y*z)*y) = x*(tz*y) \pmod n = x*(ty) \pmod n = t^2y \pmod n$ if $t^2 \equiv t \pmod n$ we have $((x*y)*z)*y = x*((y*z)*y)$ for all x,y,z in Z_n.
Now a groupoid is Moufang groupoid if $(x*y)*(z*x) = (x*(y*z))*x$ for all x,y and z in Z_n. $((x*y)*(z*x) = ty*tx \pmod n = t^2 x \pmod n$. $(x*(y*z))* x = (x* z)*x = t^2z*x = tx \pmod n$. Hence if $t^2 \equiv t \pmod n$ we see $Z_n(0,t)$ is Moufang groupoid as the Moufang identity is satisfied by all elements in $Z_n(0,t)$.
A groupoid is alternative if $(x*y)*y = x((y*y)$ and $(x*y)*y = x(x*y)$ are satisfied for all $x,y \in Z_n$.

Now $(x*y)*y = ty*y = ty \pmod n$, $x*(y*y) = x*ty = t^2y \pmod n$. So if $t^2 \equiv t \pmod n$ then it is right alternative. Also $(x*x)*y = tx*y=ty \pmod n$ and $x*(x*y) = x*ty =t^2y \pmod n$. If $t^2 \equiv t \pmod n$ then groupoid is left alternative.

Theorem 4.11 The groupoid $Z_n(t,u) \in Z(n)$ is alternative if $t^2 \equiv t \pmod n$ and $u^2 \equiv u \pmod n$; n a non prime.

Proof. A groupoid is alternative if it satisfies the following two identities $(x*y)*y = x*(y*y)$ and $(x*x)*y = x*(x*y)$ for all $x,y \in Z_n(t,u)$.
$(x*y)*y = t^2x + tuy + uy \pmod n$ and $x*(y*y) = tx + tuy + u^2y \pmod n$ if $t^2 \equiv t \pmod n$ and $u^2 \equiv u \pmod n$ then clearly $(x*y)*y = x*(y*y)$. Similarly we can prove if $t^2 \equiv t \pmod n$ and $u^2 \equiv u \pmod n$ we have the other identity to be true. Hence the result.

In the succeeding section, we propose some unsolved problems.

5. PROBLEMS

Here we propose some problems which are open and simple. Some of them need more number of theoretical techniques.

(i) Give a general formula to find the number of groupoids in $z(n)$.

(ii) Find the number of normal groupoids in $Z(n)$, $Z^{*}(n)$, $Z^{**}(n)$ and $Z^{***}(n)$

(iii) Find the number of Bol groupoids in $Z(n)$, $Z^{*}(n)$, $Z^{**}(n)$ and $Z^{***}(n)$.

(iv) How many simple groupoids are in $Z(n)$?

(v) Characterize Moufang groupoids in $Z(n)$, $Z^{*}(n)$, $Z^{**}(n)$ and $Z^{***}(n)$.

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OPTIMUM PROBABILITY OF SELECTION IN ESTIMATING FREQUENCY DISTRIBUTION FOR THE CURRENT OCCASION

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ABSTRACT. AN optimum probability of selection of the sampling units for some selected sampling designs for estimating frequency distribution for the current occasion under successive sampling approach has been proposed in this paper. Considering the sampling designs, it is observed that for estimating frequency distribution for the current occasion, use of probability proportional to size sampling under cluster as well as two-stage sampling designs are advantageous.

Keywords. Estimation, current occasion, successive sampling, error measures and probability proportional to size.

1. INTRODUCTION

The detailed study on cluster sampling and two stage sampling has been given by Murthy [3]. Singh et.al.[6] and Singh and Kathuria [5] have considered the two stage sampling under successive sampling approach for estimating the population mean. The precision of an estimate can be considerably increased by properly utilizing an auxiliary variable, x, highly correlated with the character under study, Y. An attempt has been made by Arora and Singh [1] for estimating frequency distribution for the current occasion under successive sampling for three sampling designs. For cluster and two-stage sampling designs, probability proportional to size sampling and equal probability sampling schemes have been considered. On comparing the above sampling designs on the basis of the two error measures proposed by Murthy [4], the ultimate conclusions are drawn in terms of adaptation of one of the considered design together with one of the sampling schemes for estimating the frequency distribution for the current occasion.

In this paper, an attempt has been made to propose a probability of selection in which one might use some power of the size of cluster, for cluster sampling and of primary stage unit for two-stage sampling, resulting thereby PPS and EPS as their particular cases. Mathematically the proposed probability is $\phi_i = \frac{M_i^\theta}{\Sigma M_i^\theta}$, where θ is a real number. With this probability of selection, the estimation of frequency distribution, for the current occasion using cluster and two-stage sampling under successive sampling approach, is considered. An estimate of the proportion in the h-th class of the frequency distribution (h=1,2,...k) for current occasion, for both the sample designs, is obtained, and then efficiencies of the sampling designs are compared by using the error measures E (α_1) and E (α_2) proposed by Murthy (1977), where

$$\alpha_1 = \sum_{i=1}^{k} (\hat{P}_i - P_i)^2 \qquad \text{and } \alpha_2 = \frac{1}{k} \sum_{i=1}^{k} (\hat{P}_i - P_i)^2 / P_i.$$

A design, for which both the error measures E (α_1) and E (α_2) tend to zero rapidly as the sample size increases is said to be efficient.

2. ESTIMATION OF FREQUENCY DISTRIBUTION FOR THE CURRENT OCCASION UNDER CLUSTER SAMPLING

Let there be N clusters, M_i (i = 1, 2,...,N) be the size of the i-th cluster, such that $\sum_{i=1}^{N} M_i = M$ (total number of ultimate units in the population). Select n clusters by the proposed probability $\phi_i = \frac{M_i^\theta}{\sum_{i=1}^{N} M_i^\theta}$ from N clusters on the first occasion, retain a sub-sample of n μ_c $(0 \leq \mu_c \leq 1)$ clusters from the n clusters by SRSWOR and supplement it with $(n - n\,\mu_c)$ clusters selected from (N-n) clusters by the proposed probability ϕ_i on the second occasion. Obtain the frequency distribution in the desired K classes on the basis of the ultimate units in the clusters selected at both the occasions.

With proposed probability ϕ_i, the estimator of proportion in h-th class of the frequency distribution (h=1,2...,k), takes the form

$$\hat{P}_{2hc} = \frac{\mu_c}{(1-\lambda_c^2 P_{hc}^2)} \left[\frac{\lambda_c \sigma_{12hc} (P_{1h}^{\lambda c} - P_{1h}^{\mu c})}{\sigma_{1hc}^2} + (P_{2h}^{\mu c} - P_{2h}^{\lambda c}) \right] + P_{2h}^{\lambda c}$$

where $P_{1h}^{\lambda c}$ is the estimate of the proportion of ultimate units of the unmatched clusters, on the first occasion in h-th class

$$= \frac{1}{n\lambda_c} \sum_{i=1}^{n\lambda_c} \frac{M_{1hi}}{M\Phi_i}; \qquad 0 \le \lambda_c \le 1.$$

M_{1hi} being the number of ultimate units of i-th unmatched cluster, on the first occasion, in h-th class.

$P_{2h}^{\lambda c}$ is the estimate of the proportion of the ultimate units of the unmatched clusters, on the second occasion, in h-th class

$$= \frac{1}{n\lambda_c} \sum_{i=1}^{n\lambda_c} \frac{M_{2hi}}{M\Phi_i};$$

M_{2hi} being the numbr of ultimate units of unmatched cluster, on the second occasion, in the h^{th} class.

$P_{1h}^{\mu c}$ is the estimate of the proportion of ultimate units of the matched clusters, on the frst occasion, in h-th class.

$$= \frac{1}{n\mu_c} \sum_{i=1}^{n\mu_c} \frac{M_{1hi}}{M\Phi_i};$$

M_{1hi} being the numbr of ultimate units of l-th matched cluster, on the first occasion, in the h-th class.

$P_{2h}^{\mu c}$ is the estimate of the proportion of ultimate units of the matched clusters, on the second occasion, in h-th class.

$$= \frac{1}{n\mu_c} \sum_{i=1}^{n\mu_c} \frac{M_{2hi}}{M\Phi_i}$$

M_{2hi} being the number of ultimate units of i-th matched cluster, on the second occasion, in h-th class.

$$\sigma_{1hc}^2 = \left[\sum_{i=1}^{N} \frac{w_i^2 P_{14i}^2}{\Phi_i} - P_{14}^2 \right]; \quad \text{where } w_i = \frac{M_{1i}}{M}$$

and

$$\sigma_{2hc}^2 = \left[\sum_{i=1}^{N} \frac{w_i^2 P_{2hi}^2}{\Phi_i} - P_{2h}^2 \right]$$

$$\sigma_{1hc}^2 = \left[\sum_{i=1}^{N} \frac{w_i^2 P_{1hi} P_{2hi}}{\Phi_i} - P_{1h} P_{2h} \right]$$

With proposed probability Φ_i, the error measures $E(\alpha_1)$ and $E(\alpha_2)$ are given by

$$E(\alpha_1) = \sum_{h=1}^{k} \frac{\sigma_{2hc}^2 \left[\sigma_{1hc}^2 \; \sigma_{2hc}^2 - \lambda_c \, \sigma_{12hc}^2\right]}{n \left[\sigma_{1hc}^2 \; \sigma_{2hc}^2 - \lambda_c^2 \, \sigma_{12hc}^2\right]}$$

and

$$E(\alpha_2) = \frac{1}{nk} \sum_{h=1}^{k} \frac{\sigma_{2hc}^2 \left[\sigma_{1hc}^2 \; \sigma_{2hc}^2 - \lambda_c \, \sigma_{12hc}^2\right]}{P_{2h} \left[\sigma_{1hc}^2 \; \sigma_{2hc}^2 - \lambda_c^2 \, \sigma_{12hc}^2\right]}$$

3. ESTIMATION OF FREQUENCY DISTRIBUTION FOR THE CURRENT OCCASION UNDER TWO-STAGE SAMPLING

Select a sample of n primary stage units (p.s.u.) from N p.s.u.'s by the proposed probability $\phi_i = \frac{M_i^\theta}{\sum_{i=1}^{N} M_i^\theta}$ on the first occasion. Retain a sub-sample of size n μ_t from n p.s.u.'s by SRSWOR and supplement it with $(n - n\mu_t)$ p.s.u.'s selected by the proposed probability ϕ_i from (N-n) p.s.u.'s on the second occasion. Then select a sub-sample of second stage units (s.s.u's) of size m from each of the selected cluster on both the occasions. Find the frequency distribution of s.s.u's of the matched and unmatched clusters on both the occasions in the desired K classes of the frequency distribution.

With proposed ϕ_i, the estimator of proportion in h-th class of the frequency distribution (h=1,2,...,k), takes the form

$$\hat{P}_{2ht} = \frac{\mu_t}{(1-\lambda_t^2 P_{ht}^2)} \left[\frac{\lambda_t \sigma_{12ht} (P_{1h}^{\lambda t} - P_{1h}^{\mu t})}{\sigma_{1ht}^2} + (P_{2h}^{\mu t} - P_{2h}^{\lambda t}) \right] + P_{2h}^{\lambda t}$$

where $P_{1h}^{\lambda t}$ is the estimate of the proportion of ultimate units of the unmatched p.s.u's, on the first occasion, in h-th class

$$= \frac{1}{n\lambda_t} \sum_{i=1}^{n\lambda_t} \frac{w_i}{\Phi_i} p_{1hi} ; \quad \text{where } p_{1hi} = \frac{u_{1hi}}{u_{1i}} \quad 0 \leq \lambda_t \leq 1.$$

u_{1i} being the number of s.s.u's selected from i-th unmatched p.s.u. on the first occasion and u_{1hi} being the number of selected s.s.u's from the i-th unmatched p.s.u., on the first occasion in the h-th class.

$P_{2h}^{\lambda t}$ is the estimate of the proportion of the ultimate units of the unmatched p.s.u's, on the second occasion, in the h-th class.

$$= \frac{1}{n\lambda_t} \sum_{i=1}^{n\lambda_t} \frac{w_i}{\Phi_i} p_{2hi}, \text{ where } p_{2hi} = \frac{u_{2hi}}{u_{1i}}$$

u_{2i} being the number of ultimate units selected from the i-th unmatched p.s.u., on the second occasion and u_{2hi} being the number of selected s.s.u's from the i-th unmatched p.s.u., on the second occasion, in the h-th class.

$P^{\mu t}_{1h}$ is the estimate of the proportion of the ultimate units of the matched p.s.u's, on the first occasion, in the h-th class.

$$= \frac{1}{n\mu_t} \sum_{i=1}^{n\mu_t} \frac{w_i}{\Phi_i} p_{1hi}; \text{ where } p_{2hi} = \frac{m_{1hi}}{m_{1i}}$$

m_{1i} being the number of s.s.u's selected from the i-th matched p.s.u., on the first occasion and m_{1hi} being the number of s.s.u's selectd from the i-th p.s.u., on the first occasion, in the h-th class.

$P^{\mu t}_{2h}$ is the estimate of the proportion of the ultimate units of the matched p.s.u's, on the second occasion, in h-th class.

$$= \frac{1}{n\mu_t} \sum_{i=1}^{n\mu_t} \frac{w_i}{\Phi_i} p_{2hi}; \text{ where } p_{2hi} = \frac{m_{2hi}}{m_{2i}}$$

m_{2i} being the number of ultimate units selected from the i-th matched p.s.u. on the second occasion and m_{2hi} being the number of ultimate units selected from the i-th p.s.u., on the second occasion, lying in the h-th class.

$$\sigma^2_{1ht} = \sum_{i=1}^{N} \left[\frac{w_i^2 P_{1hi}^2}{\phi_i} - P_{1h}^2 \right] + \left[\sum_{i=1}^{N} \frac{w_i^2 (M_i - m) P_{1hi} (1 - P_{1hi})}{m(M_i - 1)\phi_i} \right]$$

$$\sigma^2_{2ht} = \left[\sum_{i=1}^{N} \frac{w_i^2 P^2_{2hi}}{\phi_i} - P^2_{2h}\right] + \left[\sum_{i=1}^{N} \frac{w_i^2 (M_i - m)}{m(M_i - 1)\phi_i}\right]$$

$$\sigma_{12ht} = \left[\sum_{i=1}^{N} \frac{w_i^2 P_{1hi} \; P_{2hi}}{\phi_i} - P_{1i} P_{2i}\right] + \left[\sum_{i=1}^{N} \frac{w_i^2 (M_i - m)\,(P_{12hi} - P_{1hi} \; P_{2hi})}{m(M_i - 1)\phi_i}\right]$$

With proposed probability ϕ_i, the error measures E (α_1) and E (α_2) are given by

$$E\,(\alpha_1) = \sum_{h=1}^{k} \frac{\sigma^2_{2ht}\left[\sigma^2_{1ht}\;\sigma^2_{2ht} - \lambda_t\,\sigma^2_{12ht}\right]}{n\left[\sigma^2_{1ht}\;\sigma^2_{2ht} - \lambda_t^2\,\sigma^2_{12ht}\right]}$$

and

$$E\,(\alpha_2) = \frac{1}{nk} \sum_{h=1}^{k} \frac{\sigma^2_{2ht}\left[\sigma^2_{1ht}\;\sigma^2_{2ht} - \lambda_t\,\sigma^2_{12ht}\right]}{P_{2h}\left[\sigma^2_{1ht}\;\sigma^2_{2ht} - \lambda_t^2\,\sigma^2_{12ht}\right]}$$

4. OPTIMUM VALUE OF θ IN ESTIMATING FREQUENCY DISTRIBUTION FOR RECENT OCCASION

From the expressions of E (α_1) and E (α_2) for cluster as well as two-stage sampling design, it is not possible to work out the optimum value of θ, i.e., the value of θ for which the values of the error measures are minimum. Therefore, an empirical approach has been made. For the comparison of different sampling designs, the following two sets of data were used -

Population A- Data relating to area under bajra for 36 villages of Delhi for the years 1976-77 and 1977-78, collected under high yielding variety survey programme by Indian Agricultural Statistics Research Institute (IASRI).

Population B- Data relating to area under wheat for 36 villages of Delhi for the years 1976-77 and 1977-78, collected under high yielding variety survey by I.A.S.R.I.

The values of E (α_1) and E (α_2) have been worked out for θ in the range -5 to +5 with an interval of 1. The entire range of the l has been considered with interval 0.2 and two values of sample sizes have been considered. These values of E (α_1) and E (α_2) for different values of θ under cluster and two-stage sampling designs (for populations A and B) are presented in Tables 1, 2, 3 and 4 respectively.

From these tables, the optimum value of θ appears to be around 1 for both the error measures and also for the cluster as well as two-stage sampling designs. Further, the results are observed to be consistent for both the populations.

The range of θ from 0 to 2 was further explored with an interval of .1, to go in details for arriving at the optimum q. Since the results are similar for different values of l and also for both the sample sizes, for this investigation, only one value of l (i.e. optimum) and one value of sample size have been considered. The results for cluster sampling for both the populations A and B are presented in Table 5 while the results for two-stage sampling for both the populations A and B are presented in Table 6.

From the results, it is interesting to note that the optimum value of θ is at 1, for both error measures and also for both sampling designs. Thus for estimating frequency distributions for current occasion, use of PPS sampling under cluster as well as two-stage sampling designs is advantageous

Table 1

Values of E (α_1) and E (α_2) for Different Values of θ Under Cluster Sampling Using $\phi_i = \frac{M_i^\theta}{\Sigma M_i^\theta}$ Population A

	n	θ										
		λ	-5	-4	-3	-2	-1	0	1	2	3	4	5
E(α_1)	3	0.1	.7136	.4108	.2397	.1421	.0875	.0603	.0531	.0641	.0963	.1586	.2697
	12	0.1	.1784	.1027	.0599	.0355	.0219	.0151	.0133	.0160	.0241	.0396	.0674
	3	0.3	.6292	.3638	.2139	.1284	.0805	.0566	.0505	.0606	.0898	.1458	.2453
	12	0.3	.1573	.0909	.0535	.0321	.0201	.0142	.0126	.0152	.0224	.0365	.0613
	3	0.5	.5743	.3337	.1978	.1201	.0765	.0546	.0491	.0589	.0864	.1388	.2313
	12	0.5	.1436	.0834	.0494	.0300	.0191	.0137	.0123	.0147	.0216	.0347	.0578
	3	0.7	.5489	.3209	.1918	.1177	.0759	.0547	.0493	.0592	.0866	.1383	.2292
	12	0.7	.1372	.0802	.0479	.0294	.0190	.0137	.0123	.0148	.0216	.0346	.0573
	3	0.9	.5862	.3447	.2071	.1274	.0817	.0581	.0519	.0626	.0928	.1500	.2603
	12	0.9	.1465	.0862	.0518	.0319	.0204	.0145	.0130	.0156	.0232	.0375	.0626
E(α_2)	3	0.1	.6841	.3933	.2296	.1365	.0845	.0585	.0514	.0615	.0914	.1497	.2541
	12	0.1	.1710	.0983	.0574	.0341	.0211	.0146	.0129	.0154	.0229	.0374	.0352
	3	0.3	.6079	.3512	.2067	.1245	.0785	.0555	.0493	.0586	.0859	.1386	.2325
	12	0.3	.1520	.0878	.0517	.0311	.0196	.0139	.0123	.0147	.0215	.0346	.0581
	3	0.5	.5595	.3250	.1928	.1175	.0752	.0539	.0482	.0572	.0830	.1325	.2202
	12	0.5	.1399	.0812	.0482	.0294	.0188	.0135	.0121	.0143	.0208	.0331	.0551
	3	0.7	.5403	.3154	.1885	.1160	.0749	.0540	.0484	.0575	.0833	.1323	.2186
	12	0.7	.1351	.0788	.0471	.0290	.0187	.0135	.0121	.0144	.0208	.0331	.0546
	3	0.9	.5810	.3403	.2038	.1251	.0801	.0569	.0505	.0603	.0886	.1426	.2376
	12	0.9	.1453	.0951	.0509	.0313	.0200	.0142	.0126	.0151	.0222	.0356	.0594

Table 2

Values of E (α_1) and E (α_2) for Different Values of θ Under Cluster Sampling Using $\phi_i = \frac{M_i^{\theta}}{\Sigma M_i^{\theta}}$ Population B

	n	θ											
		λ	-5	-4	-3	-2	-1	0	1	2	3	4	5
E(α_1)	3	0.1	.6890	.3392	.1972	.1158	.0702	.0475	.0418	.0515	.0788	.1309	.2219
	12	0.1	.1472	.0848	.0493	.0289	.0175	.0119	.0105	.0129	.0197	.0327	.0565
	3	0.3	.5066	.2925	.1708	.1010	.0619	.0425	.0377	.0463	.0703	.1159	.1955
	12	0.3	.1267	.0731	.0427	.0262	.0155	.0106	.0094	.0116	.0176	.0290	.0489
	3	0.5	.5590	.2601	.1527	.0910	.0565	.0393	.0352	.0432	.0650	.1063	.1784
	12	0.5	.1123	.0660	.0382	.0228	.0141	.0098	.0988	.0108	.0162	.0266	.0446
	3	0.7	.4128	.2404	.1422	.0857	.0540	.0382	.0345	.0422	.0630	.1023	.1708
	12	0.7	.1032	.0601	.0355	.0214	.0135	.0095	.0086	.0105	.0158	.0256	.0427
	3	0.9	.4204	.2472	.1482	.0908	.0581	.0414	.0374	.0457	.0683	.1107	.1845
	12	0.9	.1051	.0618	.0371	.0227	.0145	.0104	.0093	.0114	.0171	.0277	.0461
E(α_2)	3	0.1	.5736	.3294	.1911	.1120	.0680	.0463	.0412	.0512	.0789	.1314	.2238
	12	0.1	.1434	.0824	.0178	.0280	.0170	.0116	.0103	.0128	.0197	.0329	.0560
	3	0.3	.4948	.2850	.1661	.0982	.0603	.0417	.0374	.0464	.0707	.1168	.1978
	12	0.3	.1237	.0713	.0415	.0246	.0151	.0104	.0094	.0116	.0177	.0292	.0495
	3	0.5	.4402	.2545	.1492	.0890	.0554	.0389	.0352	.0435	.0657	.1076	.1811
	12	0.5	.1101	.0636	.3731	.0223	.0138	.0097	.0088	.0109	.0164	.0269	.0453
	3	0.7	.4071	.2366	.1399	.0644	.0533	.0379	.0346	.0426	.0640	.1041	.1743
	12	0.7	.1018	.0592	.0348	.0211	.0133	.0095	.0086	.0107	.0160	.0260	.0436
	3	0.9	.4185	.2455	.1463	.0898	.0573	.0409	.0373	.0460	.0693	.1127	.1886
	12	0.9	.1048	.0614	.0367	.0224	.0143	.0102	.0093	.0115	.0173	.0282	.0471

Table 3

Values of $E(\alpha_1)$ and $E(\alpha_2)$ for Different Values of θ Under Two Stage Sampling Using $\phi_i = \frac{M_i^{\theta}}{\sum M_i^{\theta}}$ Population A

	n	θ										
		λ	-5	-4	-3	-2	-1	0	1	2	3	4	5
$E(\alpha_1)$	3	0.1	.5747	.3373	.2030	.1262	.0828	.0604	.0532	.0592	.0801	.1214	.1950
	12	0.1	.1437	.0843	.0508	.0315	.0207	.0151	.0133	.0148	.0200	.0303	.0488
	3	0.3	.5378	.3171	.1922	.1206	.0801	.0591	.0523	.0580	.0775	.1157	.1833
	12	0.3	.1344	.0793	.0481	.0302	.0200	.0143	.0131	.0145	.0194	.0289	.0458
	3	0.5	.5184	.3067	.1868	.1180	.0789	.0585	.0519	.0574	.073	.1130	.1775
	12	0.5	.1296	.0767	.0467	.0295	.0197	.0146	.0130	.0144	.0191	.0282	.0444
	3	0.7	.5191	.3075	.1876	.1186	.0793	.0588	.0521	.0577	.0768	.1137	.1783
	12	0.7	.1298	.0769	.0469	.0297	.0198	.0147	.0130	.0144	.0192	.0284	.0446
	3	0.9	.5639	.3268	.1981	.1241	.0820	.0602	.0530	.0590	.0795	.1194	.1898
	12	0.9	.1385	.0817	.0495	.0310	.0205	.0150	.0133	.0148	.0199	.0299	.0474
$E(\alpha_2)$	3	0.1	.5751	.3381	.2043	.1279	.0848	.0624	.0549	.0603	.0800	.1193	.1897
	12	0.1	.1438	.0845	.0511	.0320	.0212	.0156	.0137	.0151	.0200	.0298	.0474
	3	0.3	.5426	.3205	.1950	.1232	.0825	.0613	.0541	.0592	.0777	.1143	.1794
	12	0.3	.1357	.0801	.0488	.0308	.0206	.0153	.0135	.0148	.0194	.0286	.0448
	3	0.5	.5261	.3117	.1905	.1210	.0816	.0609	.0538	.0588	.0768	.1120	.1743
	12	0.5	.1315	.0779	.0476	.0303	.0204	.0152	.0135	.0147	.0192	.0280	.0456
	3	0.7	.5278	.3129	.1914	.1217	.0820	.0611	.0540	.0590	.0771	.1127	.1751
	12	0.7	.1320	.0782	.0479	.0304	.0205	.0153	.0135	.0148	.0193	.0282	.0438
	3	0.9	.5589	.3300	.2006	.1263	.0842	.0622	.0548	.0601	.0795	.1177	.1853
	12	0.9	.1397	.0825	.0501	.0316	.0210	.0156	.0137	.0150	.0199	.0294	.0463

Table 4

Values of $E(\alpha_1)$ and $E(\alpha_2)$ for Different Values of θ Under Two Stage Sampling Using $\phi_i = \frac{M_i^\theta}{\Sigma M_i^\theta}$ Population B

	n	θ										
		λ	-5	-4	-3	-2	-1	0	1	2	3	4	5
$E(\alpha_2)$	3	0.1	.3254	.3093	.1867	.1163	.0765	.0559	.0490	.0540	.0720	.1074	.1696
	12	0.1	.1313	.0773	.0467	.0291	.0191	.0140	.0122	.0135	.0180	.0268	.0424
	3	0.3	.4923	.2909	.1767	.1110	.0737	.0543	.0470	.0523	.0688	.1011	.1578
	12	0.3	.1231	.0727	.0442	.0277	.0184	.0136	.0119	.0131	.0172	.0253	.0394
	3	0.5	.4754	.2817	.1717	.1085	.0725	.0537	.0473	.0516	.0674	.0980	.1516
	12	0.5	.1188	.0704	.0429	.0271	.0181	.0134	.0118	.0129	.0168	.0245	.0379
	3	0.7	.4771	.2829	.1727	.1092	.0730	.0540	.0475	.0519	.0678	.0985	.1619
	12	0.7	.1193	.0707	.0432	.0273	.0182	.0135	.0119	.0130	.0170	.0246	.0380
	3	0.9	.5092	.3009	.1826	.1145	.0758	.0556	.0488	.0536	.0710	.1047	.1632
	12	0.9	.1273	.0762	.0457	.0286	.0189	.0139	.0122	.0134	.0177	.0262	.0408
$E(\alpha_2)$	3	0.1	.5262	.3096	.1870	.1168	.0771	.0567	.0499	.0551	.0734	.1094	.1729
	12	0.1	.1315	.0774	.4670	.0292	.0193	.0142	.0125	.0138	.0183	.0273	.0432
	3	0.3	.4954	.2926	.1778	.1119	.0746	.0553	.0488	.0536	.0704	.1034	.1614
	12	0.3	.1239	.0731	.0444	.0280	.0187	.0138	.0122	.0134	.0176	.0258	.0404
	3	0.5	.4800	.2842	.1733	.1097	.0735	.0547	.0484	.0529	.0691	.1005	.1555
	12	0.5	.1200	.0710	.0433	.0274	.0184	.0137	.0121	.0132	.0173	.0251	.0389
	3	0.7	.4819	.2855	.1742	.1103	.0740	.0550	.0486	.0532	.0695	.1010	.1559
	12	0.7	.1205	.0714	.0436	.0276	.0185	.0138	.0122	.0133	.0174	.0253	.0380
	3	0.9	.5119	.3022	.1834	.1152	.0765	.0564	.0497	.0548	.0725	.1069	.1669
	12	0.9	.1280	.0755	.0458	.0288	.0191	.0141	.0124	.0137	.0181	.0267	.0417

Table 5

Values of $E(\alpha_1)$ and $E(\alpha_2)$ for Cluster Sampling for optimum value of λ for Populations A and B (n=20)

Population A			Population B		
θ	$E(\alpha_1)$	$E(\alpha_2)$	θ	$E(\alpha_1)$	$E(\alpha_2)$
0	.01367	.01348	0	.00955	.00948
0.1	.01336	.01318	0.1	.00933	.00926
0.2	.01308	.01291	0.2	.00914	.00903
0.3	.01285	.01267	0.3	.00897	.00893
0.4	.01266	.01248	0.4	.00884	.00880
0.5	.01250	.01232	0.5	.00873	.00871
0.6	.01238	.01219	0.6	.00866	.00864
0.7	.01230	.01211	0.7	.00860	.00859
0.8	.01226	.01206	0.8	.00858	.00858
0.9	.01225	.01204	0.9	.00859	.00860
1.0	.01228	.01206	1.0	.00862	.00864
1.1	.01235	.01212	1.1	.00868	.00871
1.2	.01246	.01221	1.2	.00877	.00881
1.3	.01261	.01234	1.3	.00889	.00894
1.4	.01279	.01251	1.4	.00904	.00909
1.5	.01301	.01271	1.5	.00922	.00928
1.6	.01327	.01295	1.6	.00942	.00949
1.7	.01358	.01323	1.7	.00966	.00974
1.8	.01392	.01355	1.8	.00992	.01001
1.9	.01431	.01391	1.9	.01022	.01032
2.0	.01473	.01430	2.0	.01055	.01066

Table 6

Values of $E(\alpha_1)$ and $E(\alpha_2)$ for Two-Stage Sampling for optimum value of λ for Populations A and B (n=20)

	Population A			Population B	
θ	$E(\alpha_1)$	$E(\alpha_2)$	θ	$E(\alpha_1)$	$E(\alpha_2)$
0	.01463	.01522	0	.01342	.01368
0.1	.01432	.01491	0.1	.01313	.01340
0.2	.01405	.01462	0.2	.01288	.01314
0.3	.01380	.01437	0.3	.01265	.01291
0.4	.01359	.01415	0.4	.01245	.01271
0.5	.01341	.01396	0.5	.01227	.01254
0.6	.01326	.01380	0.6	.01213	.01240
0.7	.01314	.01367	0.7	.01201	.01228
0.8	.01305	.01357	0.8	.01192	.01220
0.9	.01300	.01350	0.9	.01195	.01213
1.0	.01297	.01346	1.0	.01182	.01210
1.1	.01297	.01345	1.1	.01181	.01209
1.2	.01300	.01347	1.2	.01182	.01211
1.3	.01306	.01352	1.3	.01186	.01216
1.4	.01316	.01360	1.4	.01193	.01223
1.5	.01328	.01370	1.5	.01203	.01233
1.6	.01343	.01384	1.6	.01215	.01246
1.7	.01362	.01404	1.7	.01229	.01261
1.8	.01383	.01421	1.8	.01247	.01279
1.9	.01408	.01444	1.9	.01267	.01300
2.0	.01436	.01470	2.0	.01290	.01323

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FRACTIONAL INTEGRATION OF SISTER CELINE'S POLYNOMIALS OF SEVERAL VARIABLES

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ABSTRACT. In the present paper, we have established four fractional integrals of two and more variables of Sister Celine's polynomials of two and more variables. Some particular cases have been discussed such as fractional integrals of Jacobi polynomials of two variables and Bessel function of two variables defined by the author.

Keywords. Fractional integrations, Sister Celine's polynomials, Orthogonal polynomials.
MSC (2001) : 26A33, 47G10, 45P05, 33C45, 33C50, 33C65

1. INTRODUCTION

Fasenmyer [4,5; see also 10, p.290] defined a polynomial known as Sister Celine's polynomials generated by

$$(1-t)^{-1}\ {}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};\frac{-4xt}{(1-t)^2}\right)=\sum_{n=0}^{\infty} f_n\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};x\right)t^n, \tag{1.1}$$

which yields

$$f_n(x)\equiv f_n\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};x\right)={}_{p+2}F_{q+2}\left(\begin{matrix}-n,n+1,a_1,\ldots,a_p\\ 1,\frac{1}{2},b_1,\ldots,b_q\end{matrix};x\right), \tag{1.2}$$

Her polynomials include many special cases mainly special Jacobi, Rice, Bateman, Pasternak polynomials. Author [39] has generalized the polynomials and given the definitions of Sister celine's polynomials of two and several variables in terms of generating function as follows -

$$\prod_{i=1}^{r}(1-t_i)^{-1-c_i}\,F^{p:p_1;\ldots;p_r}_{q:q_1;\ldots;q_r}\left(\begin{matrix}\left(a_j:\alpha'_j,\ldots,\alpha_j^{(r)}\right)_{1,p}:\left(c'_j,\gamma'_j\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r}\\ \left(b_j:\beta'_j,\ldots,\beta_j^{(r)}\right)_{1,q}:\left(d'_j,\delta'_j\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\end{matrix}:\frac{(-4x_1t_1)^{\lambda_1}}{(1-t_1)^{2\lambda_1}},\ldots,\frac{(-4x_rt_r)^{\lambda_r}}{(1-t_r)^{2\lambda_r}}\right.$$

$$=\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty} f_{n_1,\dots,n_r}\left(\begin{matrix}\left(a_j:\alpha_j',\dots,\alpha_j^{(r)}\right)_{1,p}:\left(c_j',\gamma_j'\right)_{1,p_1};\dots;\left(c_r^{(r)},\gamma_r^{(r)}\right)_{1,p_r}\\ \left(b_j:\beta_j',\dots,\beta_j^{(r)}\right)_{1,q}:\left(d_j',\delta_j'\right)_{1,q_1};\dots;\left(d_r^{(r)},\delta_r^{(r)}\right)_{1,q_r}\end{matrix};x_1^{\lambda_1},\dots,x_r^{\lambda_r}\right)t_1^{n_1},\dots,t_r^{n_r},\quad(13)$$

which gives

$$f_{n_1,\dots,n_r}\left(x_1^{\lambda_1},\dots,x_r^{\lambda_r}\right)\equiv f_{n_1,\dots,n_r}\left(\begin{matrix}\left(a_j:\alpha_j',\dots,\alpha_j^{(r)}\right)_{1,p}:\left(c_j',\gamma_j'\right)_{1,p_1};\dots;\left(c_r^{(r)},\gamma_r^{(r)}\right)_{1,p_r}\\ \left(b_j:\beta_j',\dots,\beta_j^{(r)}\right)_{1,q}:\left(d_j',\delta_j'\right)_{1,q_1};\dots;\left(d_r^{(r)},\delta_r^{(r)}\right)_{1,q_r}\end{matrix};x_1^{\lambda_1},\dots,x_r^{\lambda_r}\right)$$

$$=\prod_{i=1}^{r}\frac{(1+c_i)_{n_i}}{(n_i)!}F^{p,p_1+2;\dots;p_r+2}_{q,q_1+2;\dots;q_r+2}\left(\begin{matrix}\left(a_j:\alpha_j',\dots,\alpha_j^{(r)}\right)_{1,p}:(-n_1,\lambda_1),(1+n_1+c_1,\lambda_1),\left(c_j',\gamma_j'\right)_{1,p_1};\dots;\\ \left(b_j:\beta_j',\dots,\beta_j^{(r)}\right)_{1,q}:\left(\frac{1+c_1}{2},\lambda_1\right)\left(\frac{2+c_1}{2},\lambda_1\right)\left(d_j',\delta_j'\right)_{1,q_1};\dots;\end{matrix}\right.$$

$$\left.\begin{matrix}(-n_r,\lambda_r),(n_r+c_r+1,\lambda_r),\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r}\\ \left(\frac{1+c_r}{2},\lambda_r\right),\quad\left(\frac{2+c_r}{2}1,\lambda_r\right),\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\end{matrix};x_1^{\lambda_1},\dots,x_r^{\lambda_r}\right),\quad(1.4).$$

In (1.3) and (1.4), taking r = 2, we obtain the definition of Sister Celine's polynomials of two-variable

$$(1-t)^{-1-c}(1-h)^{-1-d}F^{p:p_1;p_2}_{q:q_1;q_2}\left(\begin{matrix}\left(a_j:\alpha_j',\alpha_j''\right)_{1,p}:\left(c_j',\gamma_j'\right)_{1,p_1};\left(c_j'',\gamma_j''\right)_{1,p_2}\\ \left(b_j:\beta_j',\beta_j''\right)_{1,q}:\left(d_j',\delta_j'\right)_{1,q_1};\left(d_1'',\delta_j''\right)_{1,q_2}\end{matrix};\frac{(-4xt)^{\lambda}}{(1-t)^{2\lambda}},\frac{(-4yh)^{\mu}}{(1-h)^{2\mu}}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f_{n,m}\left(\begin{matrix}\left(a_j:\alpha_j',\alpha_j''\right)_{1,p}:\left(c_j',\gamma_j'\right)_{1,p_1};\left(c_j'',\gamma_j''\right)_{1,p_2}\\ \left(b_j:\beta_j',\beta_j''\right)_{1,q}:\left(d_j',\delta_j'\right)_{1,q_1};\left(d_j'',\delta_j''\right)_{1,q_2}\end{matrix};x^{\lambda},y^{\mu}\right)t^n h\quad(1.5)$$

and

$$f_{n,m}(x^{\lambda},y^{\mu})=f_{n,m}\left(\begin{matrix}\left(a_j:\alpha_j',\alpha_j''\right)_{1,p}:\left(c_j',\gamma_j'\right)_{1,p_1};\left(c_j'',\gamma_j''\right)_{1,p_2}\\ \left(b_j:\beta_j',\beta_j''\right)_{1,q}:\left(d_j',\delta_j'\right)_{1,q_1};\left(d_j'',\delta_j''\right)_{1,q_2}\end{matrix};x^{\lambda},y^{\mu}\right)$$

$$= \frac{(1+c)_n (1+d)_m}{n!\; m!} F^{p:p_1+2;q_2+2}_{q:q_1+2;q_2+2}\left(\begin{matrix} \left(a_j:\alpha'_j,\alpha''_j\right)_{1,p} & :(-n,\lambda),(1+n+c,\lambda)\left(c'_j,\gamma'_j\right)_{1,p_1}; \\ \left(b_j:\beta'_j,\beta''_j\right)_{1,q} & :\left(\frac{1+c}{2},\lambda\right),\left(\frac{2+c}{2},\lambda\right)\left(d'_j,\delta'_j\right)_{1,q_1}; \end{matrix} \right.$$

$$\left. \begin{matrix} (-m,\mu),(1+m+d,\mu),\left(c''_j,\gamma''_j\right)_{1,p_2} \\ \left(\frac{1+d}{2},\mu\right),\left(\frac{2+d}{2},\mu\right),\left(d''_j,\delta''_j\right)_{1,q_2} \end{matrix} ;x^{\lambda},y^{\mu} \right), \qquad (1.6).$$

If we put $p = q = m = c = d = 0$ and also $\lambda, \mu, \gamma's, \delta's$ to be unity, we obtain the Sister Celine's polynomials of single variables (1.1) and (1.2).

The series involved in relation (1.3), (1.4); (1.5),(1.6) are well-known generalized Lauricella function of r-variables and 2 - variables respectively defined by Srivastava and Daoust [19; see also 22, p.37] :

$$F^{p:p_1;\dots;p_r}_{q:q_1;\dots;q_r}(x_1,\dots,x_r) \equiv F^{p:p_1;\dots;p_r}_{q:q_1;\dots;q_r}\left(\begin{matrix} \left(a_j:\alpha'_j,\dots,\alpha^{(r)}_j\right)_{1,p} :\left(c'_j,\gamma'_j\right)_{1,p_1};\dots;\left(c^{(r)}_j,\gamma^{(r)}_j\right)_{1,p_r} \\ \left(b_j:\beta'_j,\dots,\beta^{(r)}_j\right)_{1,q} :\left(d'_j,\delta'_j\right)_{1,q_1};\dots;\left(d^{(r)}_j,\delta^{(r)}_j\right)_{1,q_r} \end{matrix} ;x_1,\dots,x_r \right)$$

$$= \sum_{m_1=0}^{\infty} . \sum_{m_r=0}^{\infty} .\Omega\,(m_1,\dots,m_r)\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_1^{m_r}}{m_r!}\,, \qquad (1.7)$$

where, for convenience

$$\Omega(m_1,\dots,m_r) = \frac{\prod_{j=1}^{p}(a_j)_{m_1\alpha'_j+\dots+m_r\alpha^{(r)}_j}\prod_{j=1}^{p_1}(c'_j)_{m_1\gamma'_j}\cdots\prod_{j=1}^{p_r}(c^{(r)}_j)_{m_r\gamma^{(r)}_j}}{\prod_{j=1}^{q}(b_j)_{m_1\beta'_j+\dots+m_r\beta^{(r)}_j}\prod_{j=1}^{q_1}(d'_j)_{m_1\delta'_j}\cdots\prod_{j=1}^{q_r}(d^{(r)}_j)_{m_r\delta^{(r)}_j}},$$

the $a's$, $b's$, $c's$, $d's$ are complex numbers and coefficients

$$\alpha^{(k)}_j, j=1,\dots,p; \beta^{(k)}_j, j=1,\dots,q; \gamma^{(k)}_j, j=1,\dots,p^{(k)}_k; \delta^{(k)}_j, j=1,\dots,q^{(k)}_k; (k=1,\dots,r) \qquad (1.8)$$

are real and positive.

The multiple series (1.7) and it's special case when r = 2 converge absolutely [20, section 5 (p.157-158), section 3,4 (p.153-157); 2, section 3.7; 3, section 1.4].

When each of the positive numbers listed in (1.8) is equated to unity, the generalized Lauricella series (1.7) reduces to a multi variable Kampe-de-Feriet series (function).

The computation of fractional integration (and fractional derivative) of special functions of one and more variables is important, in the point of view of usefulness of the results, such as in the evaluation of series & integrals [8, 9, 41], the derivation of generating functions [21] and solution of differential & integral equations [6, 7, 8, 9, 18, 24]. Motivated by these and many other avenues of applications, many workers obtained several integral (derivative) formulae involving different special functions [17,23,25-36].

Author [40] introduced two integral operators involving Kampe-de-Feriet function of several variables. On specializing parameters involved in Kampe-de-Feriet function, we obtain another two integral operators involving Appell's function. These integral operators contain as special cases the well-known integral operators defined by Saigo [11, see also 12-16], Riemann-Liouville, Erdelyi-Kober and Weyl fractional integral operators. These operators are defined as follows :

$$I_{x_1,\dots,x_r}^{a_1,b_1;\dots;a_r,b_r;c} f(x_1,\dots,x_r) = \frac{x_1^{-a_1-b_1}\dots x_r^{-a_r-b_r}}{\Gamma(a_1)\dots\Gamma(a_r)} \int_0^{x_1}\dots\int_0^{x_r} (x_1-t_1)^{a_1-1}\dots(x_r-t_r)^{a_r-1} .$$

$$\times F_{0:1;\dots;1}^{1:1;\dots;1}\left(\begin{matrix} -c : a_1+b_1;\dots;a_r+b_r \\ - : a_1;\dots;a_r \end{matrix} ;\left(1-\frac{t_1}{x_1}\right),\dots,\left(1-\frac{t_r}{x_r}\right)\right) f(t_1,\dots,t_r)\,dt_1\dots dt_r \quad ,(1.9).$$

$$J_{x_1,\dots,x_r}^{a_1,b_1;\dots;a_r,b_r;c} f(x_1,\dots,x_r) = \frac{1}{\Gamma(a_1)\dots\Gamma(a_r)} \int_{x_1}^{\infty}\dots\int_{x_r}^{\infty} (t_1-x_1)^{a_1-1}\dots(t_r-x_r)^{a_r-1}\, t_1^{-a_1-b_1}\dots t_r^{-a_r-b_r}$$

$$\times F_{0:1;\dots;1}^{1:1;\dots;1}\left(\begin{matrix} -c : a + b_1;\dots;a_r+b_r \\ - : a_1;\dots;a_r \end{matrix} ;\left(1-\frac{x_1}{t_1}\right),\dots,\left(1-\frac{x_r}{t_r}\right)\right) f(t_1,\dots,t_r)\,dt_1\dots dt_r \quad ,(1.10).$$

In (1.9) and (1.10) and taking r = 2, we get fractional integral operators for two-variables involving Appell's function F_2 :

$$I_{x,y}^{a,b;a',b';c} f(x,y) = \frac{x^{-a-b}y^{-a'-b'}}{\Gamma(a)\Gamma(a')}\int_0^x\int_0^y (x-t)^{a-1}(y-h)^{a-1}F_2\left(-c;a+b,a'+b';a,a';1-\frac{t}{x},1-\frac{h}{y}\right)f(t,h)\,dtdh, \tag{1.11}$$

and

$$J_{x,y}^{a,b;a',b';c} f(x,y) = \frac{1}{\Gamma(a)\Gamma(a')}\int_x^\infty .\int_y^\infty (t-x)^{a-1}(h-y)^{a'-1}t^{-a-b}h^{-a'-b'}$$
$$\times F_2\left(-c;a+b,a'+b';a,a';1-\frac{x}{t},1-\frac{y}{h}\right) f(t,h)\,dtdh, \tag{1.12}$$

where $a_i, a, a' > 0, b_i, b, b', c$ are real numbers (also valid for complex numbers). $\mathrm{Re}(a_i), \mathrm{Re}(a), \mathrm{Re}(a') > 0, b_i, b, b'$; ($i = 1, \ldots, r$).

Reductions

(i) In (1.9) and (1.10), taking , the integral operators I and J reduce to the following well-known integral operators :

$$I_{x_1,\ldots,x_r}^{a_1,-a_1;\ldots;a_r,-a_r,c} f(x_1,\ldots,x_r)$$
$$= \frac{1}{\Gamma(a_1)\ldots\Gamma(a_r)}\int_0^{x_1}\ldots\int_0^{x_r}(x_1-t_1)^{a_1-1}\ldots(x_r-t_r)^{a_r-1}f(t_1,\ldots,t_r)dt_1\ldots dt_r = R_{x_1,\ldots,x_r}^{a_1,\ldots,a_r,c} f(x_1,\ldots,x_r), \tag{1.13}$$

which is an analogue of Riemann-Liouville integral operator for r-variables.

(ii) $$I_{x_1,\ldots,x_r}^{a_r,0;\ldots;a_r,0;c} f(x_1,\ldots,x_r) = \frac{x_1^{-a_1}\ldots x_r^{-a_r}}{\Gamma(a_1)\ldots\Gamma(a_r)}\int_0^{x_1}\ldots\int_0^{x_r}(x_1-t_1)^{a_1-1}\ldots(x_r-t_r)^{a_r-1}$$

$$\left[\left(\frac{t_1}{x_1}+\ldots+\frac{t_r}{x_r}\right)-(r-1)\right]^c f(t_1,\ldots,t_r)\,dt_1\ldots dt_r = E_{x_1,\ldots,x_r}^{a_1,\ldots,a_r,c} f(x_1,\ldots,x_r), \tag{1.14}$$

which is an analogue of Erdelyi-Kober integral operator for r-variables.

$$J_{x_1,\ldots,x_r}^{a_1,-a_1;\ldots;a_r,-a_r;c} f(x_1,\ldots,x_r)$$

$$=\frac{1}{\Gamma(a_1)..\Gamma(a_r)}\int_{x_1}^{\infty}\ldots\int_{x_r}^{\infty}(t_1-x_1)^{a_1-1}\ldots(t_r-x_r)^{a_r-1}f(t_1,\ldots,t_r)dt_1..dt_r=W_{x_1,\ldots,x_r}^{a_1,\ldots a_r,c}f(x_1,\ldots,x_r), \quad (1.15)$$

which is an analogue of Weyl integral operator for r-variables.

$$J_{x_1,\ldots,x_r}^{a_1,0,\ldots,a_r,0:c} f(x_1,\ldots,x_r)=\frac{1}{\Gamma(a_1)..\Gamma(a_r)}\int_{x_1}^{\infty}\ldots\int_{x_r}^{\infty}(t_1-x_1)^{a_1-1}\ldots(t_r-t_r)^{a_r-1}t_1^{-a_1}..t_r^{-a_r}$$

$$\times\left[\left(\frac{x_1}{t_1}+\ldots+\frac{x_r}{t_r}\right)-(r-1)\right]^{c} f(t_1,\ldots,t_r)\,dt_1\ldots dt_r\ , \quad (1.16)$$

$$=K_{x_1,\ldots,x_r}^{a_1,\ldots,a_r,c} f(x_1,\ldots,x_r)\ , \quad (1.17)$$

which is an analogue of Erdelyi-Kober integral operator for r-variables.

2. FRACTIONAL INTEGRALS

In this section, we shall establish following fractional integrals of Celine's polynomials of 2 - variables and in general r-variables

$$I_{x,y}^{a,b;a',b':c}\left[x^{\alpha+n-a}y^{\beta+m-a'}f_{n,m}(x,y)\right]=\frac{x^{\alpha+n-a-b}y^{\beta+m-a'-b'}\Gamma(1+\alpha+n-a)\Gamma(1+\beta+m-a')}{\Gamma(1+\alpha+n)\Gamma(1+\beta+m)}$$

$$\times F_{q+2:q_1+2:q_2+2:0:0}^{p+1:p_1+3:p_2+3:1:1}\left(\begin{matrix}(-c:0,0,1,1),(a_j:\alpha_j',\alpha_j'',0,0)_{1,p}\\(1+\alpha+n:1,0,1,0),(1+\beta+m:0,1,0,1),(b_j:\beta_j',\beta_j'',0,0)_{1,q}\end{matrix}\ :\right.$$

$$\begin{matrix}(-n,1),(n+1,1),(1+\alpha+n-a,1),(c_j',\gamma_j')_{1,p_1};\\(\frac{1}{2},1),(1,1),(d_j',\delta_j')_{1,q_1};\end{matrix}$$

$$\left.\begin{matrix}(-m,1),(m+1,1),(1+\beta+m-a',1),(c_j'',\gamma_j'')_{1,p_2};(a+b,1);(a'+b',1);\\(\frac{1}{2},1),(1,1),(d_j'',\delta_j'')_{1,q_2};\quad -;\quad -;\end{matrix}\ x,y,1,1\right), \quad (2.1)$$

$$I^{a_1,b_1;\ldots;a_r,b_r;c}_{x_1,\ldots,x_r}\left[(x_1^{\alpha_1+n_1-a_1}\ldots x_r^{\alpha_r+n_r-a_r})f_{n_1,\ldots,n_r}(x_1,\ldots,x_r)\right]=\prod_{i=1}^{r}\frac{x_i^{\alpha_i+n_i-a_i-b_i}\,\Gamma(1+\alpha_i+n_i-a_i)}{\Gamma(1+\alpha_i+n_i)}$$

$$\times F^{p+1:p_1+3;\ldots;p_r+3;1;\ldots;1}_{q+r:q_1+2;\ldots;q_r+2;0;\ldots;0}\left(\begin{matrix}(-c:(0,\ldots,0),(1,\ldots,1)),(a_j:(\alpha'_j,\ldots,\alpha^{(r)}_j),(0,\ldots,0))_{1,p} \\ (1+\alpha_1+n_1:(1,0,\ldots,0),(1,0,\ldots,0)),\ldots,\end{matrix}\right.$$

$$\begin{matrix} & : \\ (1+\alpha_r+n_r:(0,\ldots,0,1),(0,\ldots,0,1)),(b_j:(\beta'_j,\ldots,\beta^{(r)}_j),(0,\ldots,0))_{1,q} & :\end{matrix}$$

$$\begin{matrix}(-n_1,1),(n_1+1,1),(1+\alpha_1+n_1-a_1,1),(c'_j,\gamma'_j)_{1,p_1};\ldots;(-n_r,1),(n_r+1,1),(1+\alpha_r+n_r-a_r,1), \\ (\tfrac{1}{2},1),(1,1),(d'_j,\delta'_j)_{1,q_1};\ldots;\qquad\qquad\qquad\qquad(\tfrac{1}{2},1),(1,1),\end{matrix}$$

$$\left.\begin{matrix}(c^{(r)}_j,\gamma^{(r)}_j)_{1,p_r};(a_1+b_1,1);\ldots;(a_r+b_r,1); \\ (d^{(r)}_j,\delta^{(r)}_j)_{1,q_r};\qquad —;\ldots;\qquad —;\end{matrix}(x_1,\ldots,x_r),(1,\ldots,1)\right), \qquad (2.2).$$

$$J^{a,b;a',b';c}_{x,y}\left[x^{a+b-\alpha-n}y^{a'+b'-\beta-m}f_{n,m}\left(\frac{1}{x},\frac{1}{y}\right)\right]=\frac{x^{a-\alpha-n}y^{a'-\beta-m}\,\Gamma(\alpha+n-a)\,\Gamma(\beta+m-a')}{\Gamma(\alpha+n)\,\Gamma(\beta+m)}$$

$$\times F^{p+1:p_1+3:q_1+3:1:1}_{q+2:q_1+2:q_2+2:0:0}\left(\begin{matrix}(-c:0,0,1,1),(a_j:\alpha'_j,\alpha''_j,0,0)_{1,p} & : \\ (\alpha+n:1,0,1,0),(\beta+m:0,1,0,1),(b_j:\beta'_j,\beta''_j,0,0)_{1,q} & :\end{matrix}\right.$$

$$\begin{matrix}(-n,1),(n+1,1),(\alpha+n-a,1),(c'_j,\gamma'_j)_{1,p_1};(-m,1)(m+1,1),(\beta+m-a',1),(c''_j,\gamma''_j)_{1,p_2}; \\ (\tfrac{1}{2},1),(1,1),(d'_j,\delta'_j)_{1,q_1};\qquad\qquad\qquad(\tfrac{1}{2},1),(1,1),(d''_j,\delta''_j)_{1,q_2};\end{matrix}$$

$$\left.\begin{matrix}(a+b,1);(a'+b',1); \\ —;\qquad —;\end{matrix}x,y,1,1\right), \qquad (2.3).$$

$$J^{a_1,b_1;\ldots;a_r,b_r;c}_{x_1,\ldots,x_r}\left[(x_1^{a_1+b_1-\alpha_1-n_1}\ldots x_r^{a_r+b_r-\alpha_r-n_r})f_{n_1,\ldots,n_r}(x_1,\ldots,x_r)\right]=\prod_{i=1}^{r}\frac{x_i^{a_i-\alpha_i-n_i}\,\Gamma(\alpha_i+n_i-a_i)}{\Gamma(\alpha_i+n_i)}\ .$$

$$\times F^{p+1:p_1+3;\ldots;p_r+3;1;\ldots;1}_{q+r:q_1+2;\ldots;q_r+2;0;\ldots;0}\left(\begin{matrix}(-c:(0,\ldots,0),(1,\ldots,1))(a_j:(\alpha'_j,\ldots,\alpha^{(r)}_j),(0,\ldots,0))_{1,p} \\ (\alpha_1+n_1:(1,0,\ldots,0),(1,0,\ldots,0)),\ldots,(\alpha_r+n_r:(0,\ldots,0,1),(0,\ldots,0,1)),\end{matrix}\right.$$

$$\begin{matrix} & :(-n_1,1),(n_1+1,1),(\alpha_1+n_1-a_1,1),(c'_j,\gamma'_j)_{1,p_1};\ldots; \\ (b_j:(\beta'_j,\ldots,\beta^{(r)}_j),(0,\ldots,0))_{1,q}: & (\tfrac{1}{2},1),(1,1),(d'_j,\delta'_j)_{1,q_1};\ldots;\end{matrix}$$

$$\left.\begin{array}{l} (-n_r,1),(n_r+1,1),(\alpha_r+n_r-a_r,1),(c_j^{(r)},\gamma_j^{(r)})_{1,p_r};(a_1+b_1,1);\ldots;(a_r+b_r,1);\\ (\tfrac{1}{2},1),(1,1),(d_j^{(r)},\delta_j^{(r)})_{1,q_r};\qquad -;\ldots;\qquad -;\end{array}\left(\frac{1}{x_1},\ldots,\frac{1}{x_r}\right),(1,\ldots,1)\right) \quad (2.4)$$

provided all the above series exist.

Proof of (2.1) Taking LHS, using the definition of fractional integration of fractional integral of two-variable (1.11) and Celine's polynomials of two-variables (1.6) with $c = d = 0, \lambda = \mu = 1$; changing the order of summations and integration, we get

$$\frac{x^{-a-b}\,y^{-a'-b'}}{\Gamma(a)\,\Gamma(a')}\sum_{m_1=0}^{n}\sum_{m_2=0}^{m}\sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{m_1\alpha'_j+m_2\alpha''_j}\prod_{j=1}^{p_1}(c'_j)_{m_1\gamma'_j}\prod_{j=1}^{p_2}(c''_j)_{m_2\gamma''_j}}{\prod_{j=1}^{q}(b_j)_{m_1\beta'_j+m_2\beta''_j}\prod_{j=1}^{q_1}(d'_j)_{m_1\delta'_j}\prod_{j=1}^{q_2}(d''_j)_{m_2\delta''_j}}$$

$$\times\frac{(-n)_{m_1}(n+1)_{m_1}(-m)_{m_2}(m+1)_{m_2}}{(\frac{1}{2})_{m_1}(1)_{m_1}(\frac{1}{2})_{m_2}(1)_{m_2}\,m_1!m_2!}\,\frac{(-c)_{s_1+s_2}(a+b)_{s_1}(a'+b')_{s_2}}{s_1!s_2!(a)_{s_1}(a')_{s_2}}\,x^{a-1}y^{a'-1}$$

$$\times\int_0^x\int_0^y\left(1-\frac{t}{x}\right)^{a+s_1-1}t^{\alpha+n+m_1-a}\left(1-\frac{h}{y}\right)^{a'+s_2-1}h^{\beta+m+m_2-a'}\,dt\,dh\,.$$

On putting $t = xu, h = yv$ and using the well-known definition of Beta function [10; eq.(1), p.18], we arrived at

$$\frac{x^{\alpha+n-a-b}\,y^{\beta+m-a'-b'}\,\Gamma(1+\alpha+n-a)\,\Gamma(1+\beta+m-a')}{\Gamma(1+\alpha+n)\,\Gamma(1+\beta+m)}$$

$$\times\sum_{m_1=0}^{n}\sum_{m_2=0}^{m}\sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{m_1\alpha'_j+m_2\alpha''_j}\prod_{j=1}^{p_1}(c'_j)_{m_1\gamma'_j}\prod_{j=1}^{p_2}(c''_j)_{m_2\gamma''_j}}{\prod_{j=1}^{q}(b_j)_{m_1\beta'_j+m_2\beta''_j}\prod_{j=1}^{q_1}(d'_j)_{m_1\delta'_j}\prod_{j=1}^{q_2}(d''_j)_{m_2\delta''_j}}$$

$$\times \frac{(-n)_{m_1}(n+1)_{m_1}(-m)_{m_2}(m+1)_{m_2}(1+\alpha+n-a)_{m_1}(1+\beta+m-a')_{m_2}(-c)_{s_1+s_2}(a+b)_{s_1}(a'+b')_{s_2}}{\left(\frac{1}{2}\right)_{m_1}(1)_{m_1}\left(\frac{1}{2}\right)_{m_2}(1)_{m_2}\; m_1!\, m_2!\, s_1!\, s_2!\,(1+\alpha+n)_{m_1+s_1}(1+\beta+m)_{m_2+s_2}}$$

$$\times x^{m_1} y^{m_2} .$$

Now using the definition of generalized Lauricella function (1.7) with r=2, we get the required result. Similarly the result (2.2) can easily be obtained.

Proof of (2.3) Taking LHS, using the definition of fractional integral of two-variable (1.12) and Celine's polynomials of two-variables (1.6) with $\lambda = \mu = 1, c = d = 0$; changing the order of summations and integration, we get

$$\frac{1}{\Gamma(a)\,\Gamma(a')}\sum_{m_1=0}^{n}\sum_{m_2=0}^{m}\sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{m_1\alpha'_j+m_2\alpha''_j}\prod_{j=1}^{p_1}(c'_j)_{m_1\gamma'_j}\prod_{j=1}^{p_2}(c''_j)_{m_2\gamma''_j}}{\prod_{j=1}^{q}(b_j)_{m_1\beta'_j+m_2\beta''_j}\prod_{j=1}^{q_1}(d'_j)_{m_1\delta'_j}\prod_{j=1}^{q_2}(d''_j)_{m_2\delta''_j}}$$

$$\times \frac{(-n)_{m_1}(n+1)_{m_1}(-m)_{m_2}(m+1)_{m_2}(-c)_{s_1+s_2}(a+b)_{s_1}(a'+b')_{s_2}}{(\frac{1}{2})_{m_1}(1)_{m_1}(\frac{1}{2})_{m_2}(1)_{m_2}(a)_{s_1}(a')_{s_2}\;\; m_1!m_2!\;\; s_1!s_2!}$$

$$\times \int_x^{\infty}\int_y^{\infty}(t-x)^{a+s_1-1}t^{-\alpha-n-m_1-s_1}(h-y)^{a'+s_2-1}h^{-\beta-m-m_2-s_2}\,dt\,dh .$$

On substitution $t-x=u, h-y=v$ and again $u=x\theta, v=y\theta$ and evaluating inner integrals with help of definition of Beta function [1,eq. (2.40), p.83], we obtain

$$\frac{x^{\alpha+\alpha-n}y^{a'-\beta-m}\,\Gamma(\alpha+n-a)\,\Gamma(\beta+m-a')}{\Gamma(\alpha+n)\;\Gamma(\beta+m)}$$

$$\times\sum_{m_1=0}^{n}\sum_{m_2=0}^{m}\sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{m_1\alpha'_j+m_2\alpha''_j}\prod_{j=1}^{p_1}(c'_j)_{m_1\gamma'_j}\prod_{j=1}^{p_2}(c''_j)_{m_2\gamma''_j}}{\prod_{j=1}^{q}(b_j)_{m_1\beta'_j+m_2\beta''_j}\prod_{j=1}^{q_1}(d'_j)_{m_1\delta'_j}\prod_{j=1}^{q_2}(d''_j)_{m_2\delta''_j}}$$

$$\times \frac{(-n)_{m_1}(n+1)_{m_1}(\alpha+n-a)_{m_1}(-m)_{m_2}(m+1)_{m_2}(\beta+m-a')_{m_2}(-c)_{s_1+s_2}(a+b)_{s_1}(a'+b')_{s_2}}{\left(\frac{1}{2}\right)_{m_1}(1)_{m_1}\left(\frac{1}{2}\right)_{m_2}(1)_{m_2}\,(\alpha+n)_{m_1+s_1}(\beta+m)_{m_2+s_2}\,m_1!\,m_2!\,s_1!\,s_2!}$$

$$\times x^{-m_1} y^{-m_2} .$$

Now using the definition of generalized Lauricella function (1.7) with r =2 two variables, we get the desired result. Similarly the result (2.4) can easily be obtained.

3. APPLICATIONS

As the results (2.1) to (2.4) involving Celine's polynomials are of most generalized character in terms of parameters as well as variables, therefore specializing both or any one, we can obtain a wide range of results involving simpler special functions. To save the space, we confined to ourselves to two variables only and discuss few main results which have great potential to explore in many ways. The applications of these formulae may provide potentially useful generalizations of known results in theory of polynomials and fractional integrations.

(i) In (2.1), taking $a+b=a'+b'=0$ or $b=b'=0$, we obtain respectively Reimann-Liouville $R_{x,y}^{a,a'}$ and Erdelyi-Kober $E_{x,y}^{a,a'}$ fractional integrals of Celine's polynomials of two variables.

$$R_{x,y}^{a,a'}\left[x^{\alpha+n-a}y^{\beta+m-a'}f_{n,m}(x,y)\right]=\frac{x^{\alpha+n}y^{\beta+m}\,\Gamma(1+\alpha+n-a)\,\Gamma(1+\beta+m-a')}{\Gamma(1+\alpha+n)\,\Gamma(1+\beta+m)}$$

$$\times f_{n,m}\left(\begin{matrix}(a_j:\alpha'_j,\alpha''_j)_{1,p}:(1+\alpha+n-a,1),(c'_j,\gamma'_j)_{1,p_1};(1+\beta+m-a',1),(c''_j,\gamma''_j)_{1,p_2}\\(b_j:\beta'_j,\beta''_j)_{1,q}:\quad(1+\alpha+n,1),(d'_j,\delta'_j)_{1,q_1};\quad(1+\beta+m,1),(d''_j,\delta''_j)_{1,q_2}\end{matrix}\;x,y\right),\quad(3.1)$$

$$E_{x,y}^{a,a'}\left[x^{\alpha+n-a}y^{\beta+m-a'}f_{n,m}(x,y)\right]=\frac{x^{\alpha+n-a}y^{\beta+m-a'}\,\Gamma(1+\alpha+n-a)\,\Gamma(1+\beta+m-a')}{\Gamma(1+\alpha+n)\,\Gamma(1+\beta+m)}$$

$$\times F^{\,p+1:p_1+3:p_2+3:1:1}_{\,q+2:q_1+2:q_2+2:0:0}\left(\begin{matrix}(-c:0,0,1,1),(a_j:\alpha'_j,\alpha''_j)_{1,p} & :\\(1+\alpha+n:1,0,1,0),(1+\beta+m:0,1,0,1),(b_j:\beta'_j,\beta''_j)_{1,q} & :\end{matrix}\right.$$

$$\begin{matrix}(-n,1),(n+1,1),(1+\alpha+n-a,1),(c'_j,\gamma'_j)_{1,p_1};(-m,1),(m+1,1),(1+\beta+m-a',1),\\ (\tfrac{1}{2},1),(1,1),(d'_j,\delta'_j)_{1,q_1}; \qquad\qquad\qquad\qquad (\tfrac{1}{2},1),(1,1).\end{matrix}$$

$$\left.\begin{matrix}(c''_j,\gamma''_j)_{1,p_2};(a,1);(a',1);\\ (d''_j,\delta''_j)_{1,q_2};\ -;\ -;\end{matrix}\ x,y,1,1\right), \qquad (3.2).$$

(ii) In (2.3) putting same as in (i), we get respectively Weyl $W^{a,a'}_{x,y}$ and Erdelyi-Kober $K^{a,a'}_{x,y}$ fractional integrals of Celine's polynomials of two variables.

$$W^{a,a'}_{x,y}\left[x^{-\alpha-n}y^{-\beta-m}f_{n,m}\left(\frac{1}{x},\frac{1}{y}\right)\right]=\frac{x^{a-\alpha-n}y^{a'-\beta-m}\,\Gamma(\alpha+n-a)\,\Gamma(\beta+m-a')}{\Gamma(\alpha+n)\,\Gamma(\beta+m)}$$

$$\times f_{n,m}\left(\begin{matrix}(a_j:\alpha'_j,\alpha''_j)_{1,p}:(\alpha+n-a,1),(c'_j,\gamma'_j)_{1,p_1};(\beta+m-a',1),(c''_j,\gamma''_j)_{1,p_2};\\ (b_j:\beta'_j,\beta''_j)_{1,q}:\quad(\alpha+n,1),(d'_j,\delta'_j)_{1,q_1};\quad(\beta+m,1),(d''_j,\delta''_j)_{1,q_2};\end{matrix}\ \frac{1}{x},\frac{1}{y}\right), (3.3)$$

$$K^{a,a'}_{x,y}\left[x^{a-\alpha-n}y^{a'-\beta-m}f_{n,m}\left(\frac{1}{x},\frac{1}{y}\right)\right]=\frac{x^{a-\alpha-n}y^{a'-\beta-m}\,\Gamma(\alpha+n-a)\Gamma(\beta+m-a')}{\Gamma(\alpha+n)\Gamma(\beta+m)}$$

$$\times F^{p+1:p_1+3;p_2+3;1;1}_{q+2:q_1+2;q_2+2;0;0}\left(\begin{matrix}(-c:0,0,1,1),(a_j:\alpha'_j,\alpha''_j)_{1,p} \qquad\qquad :\\ (\alpha+n:1,0,1,0),(\beta+m:0,1,0,1),(b_j:\beta'_j,\beta''_j)_{1,q}:\end{matrix}\right.$$

$$\begin{matrix}(-n,1),(n+1,1),(\alpha+n-a,1),(c'_j,\gamma'_j)_{1,p_1};(-m,1),(m+1,1),(\beta+m-a',1),\\ (\tfrac{1}{2},1),(1,1),(d'_j,\delta'_j)_{1,q_1}; \qquad\qquad\qquad\qquad (\tfrac{1}{2},1),(1,1),\end{matrix}$$

$$\left.\begin{matrix}(c''_j\gamma''_j)_{1,p_2};(a,1);(a',1);\\ (d''_j,\delta''_j)_{1,q_2};\ -\ ;\ -\ ;\end{matrix}\ \frac{1}{x},\frac{1}{y},1,1\right), \qquad (3.4).$$

(iii)In (2.1) and (2.3), putting $p=1, a_1=-n; q=0; p_1=3, c'_1=1+\alpha_1+\beta_1+n, c'_2=\frac{1}{2}$, $c'_3=1; q_1=3, d'_1=1+\alpha_1, d'_2=-n, d'_3=n+1; p_2=3, c''_1=1+\alpha_2+\beta_2+n, c''_2=\frac{1}{2}, c''_3=1; q_2=3.$

$d_1'' = 1+\alpha_2, d_2'' = -m, d_3'' = m+1; \alpha's, \beta's, \gamma's, \delta's$ to be unity in Celine's polynomials, we get fractional integrals of Jacobi polynomials of two variables [37].

$$I_{x,y}^{a,b;a',b',c}\left[x^{\alpha+n-a}y^{\beta+m-a'}P_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}(1-2x,1-2y)\right]$$

$$= \frac{x^{\alpha+n-a-b}y^{\beta+m-a'-b'}(1+\alpha_1)_n(1+\alpha_2)_n\,\Gamma(1+\beta+m\text{-}a')\,\Gamma(1+\alpha+n-a)}{(n!)^2\,\Gamma(1+\alpha+n)\,\Gamma(1+\beta+m)}$$

$$\times F_{2:1;1:0;0}^{2:2;2:1;1}\left(\begin{array}{l} (-n:1,1,0,0),(-c:0,0,1,1):(1+\alpha_1+\beta_1+n,1),(1+\alpha+n-a,1); \\ (1+\alpha+n:1,0,1,0),(1+\beta+m:0,1,0,1):\qquad\qquad (1+\alpha_1,1); \end{array}\right.$$

$$\left.\begin{array}{l} (1+\alpha_2+\beta_2+n,1),(1+\beta+m-a',1);(a+b,1);(a'+b',1); \\ (1+\alpha_2,1)\qquad\qquad\qquad\qquad —;\qquad\qquad —; \end{array} x,y,1,1\right) \quad (3.5)$$

$$J_{x,y}^{a,b;a',b';c}\left[x^{\alpha+b-\alpha-n}y^{a'+b'-\beta-m}P_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}\left(\frac{x-2}{x},\frac{y-2}{y}\right)\right]$$

$$= \frac{x^{a-\alpha-n}y^{\alpha'-\beta-m}(1+\alpha_1)_n(1+\alpha_2)_n\,\Gamma(\alpha+n\text{-}a)\,\Gamma(\beta+m-a')}{(n!)^2\,\Gamma(\alpha+n)\,\Gamma(\beta+m)}$$

$$\times F_{2:1;1:0;0}^{2:2;2:1;1}\left(\begin{array}{l} (-n:1,1,0,0),(-c:0,0,1,1):(1+\alpha_1+\beta_1+n,1),(\alpha+n-a,1); \\ (\alpha+n:1,0,1,0),(\beta+m:0,1,0,1):\qquad\qquad (1+\alpha_1,1); \end{array}\right.$$

$$\left.\begin{array}{l} (1+\alpha_2+\beta_2+n,1),(\beta+m-a',1);(a+b,1);(a'+b',1); \\ (1+\alpha_2,1)\qquad\qquad\qquad ;\qquad —;\qquad\qquad —; \end{array} \frac{1}{x},\frac{1}{y},1,1\right), \quad (3.6)$$

(iv) In (2.1) and (2.3), setting $p=0; q=0; p_1=3. c_1' = 1+\alpha_1+\beta_1+n. c_2' = \frac{1}{2}. c_3' = 1; q_1 = 2;$ $d_1' = n+1,\ d_2' = 1+\alpha_1;\quad p_2 = 3,\quad c_1'' = 1+\alpha_2+\beta_2+m,\ c_2'' = \frac{1}{2}, c_3'' = 1;\ q_2 = 2, d_1'' = m+1,$ $d_2'' = 1+\alpha_2; \alpha's, \beta's, \gamma's, \delta's$ to be unity in Celine's polynomials, we obtain fractional integrals of product of Jacobi polynomials [37].

$$I_{x,y}^{a,b;a',b';c}\left[x^{\alpha+n-a}y^{\beta+m-a'}P_n^{(\alpha_1,\beta_1)}(1-2x)P_m^{(\alpha_2,\beta_2)}(1-2y)\right]$$

$$=\frac{x^{\alpha+n-a-b}y^{\beta+m-a'-b'}(1+\alpha_1)_n(1+\alpha_2)_m\,\Gamma(1+\alpha+n\text{-}a)\,\Gamma(1+\beta+m\text{-}a')}{n!\,m!\,\Gamma(1+\alpha+n)\,\Gamma(1+\beta+m)}$$

$$\times F_{2:1;1;0;0}^{1:3;3;1;1}\left(\begin{array}{l}(-c:0,0,1,1)\qquad\qquad :(-n,1),(1+\alpha_1+\beta_1+n,1),(1+\alpha+n-a,1);\\(1+\alpha+n:1,0,1,0),(1+\beta+m:0,1,0,1):(1+\alpha_1,1)\end{array}\right.$$

$$\left.\begin{array}{l}(-m,1),(1+\alpha_2+\beta_2+m,1),(1+\beta+m-a',1);(a+b,1);(a'+b',1);\\(1+\alpha_2,1)\qquad\qquad\qquad\qquad\qquad\qquad ;\quad -;\qquad -;\end{array}x,y,1,1\right),\ (3.7).$$

$$J_{x,y}^{a,b,a',b';c}\left[x^{a+b-\alpha-n}y^{a'+b'-\beta-m}P_n^{(\alpha_1,\beta_1)}\left(\frac{x-2}{x}\right)P_m\left(\frac{y-2}{y}\right)\right]$$

$$=\frac{x^{a-\alpha-n}y^{a'-\beta-m}(1+\alpha_1)_n(1+\alpha_2)_m\,\Gamma(\alpha+n-a)\,\Gamma(\beta+m-a')}{n!\,m!\quad\Gamma(\alpha+n)\quad\Gamma(\beta+m)}$$

$$F_{2:1:1;0;0}^{1:3;3;1;1}\left(\begin{array}{l}(-c:0,0,1,1)\qquad\qquad :(-n,1),(1+\alpha_1+\beta_1+n,1),(\alpha+n-a,1);\\(\alpha+n:1,0,1,0),(\beta+m:0,1,0,1):(1+\alpha_1,1)\qquad\qquad\qquad\qquad ;\end{array}\right.$$

$$\left.\begin{array}{l}(-m,1),(1+\alpha_2+\beta_2+m,1),(\beta+m-a',1);(a+b,1);(a'+b',1);\\(1+\alpha_2,1)\qquad\qquad\qquad\qquad\qquad\qquad ;\quad -;\qquad -;\end{array}\frac{1}{x},\frac{1}{y},1,1\right),\quad (3.8).$$

It is worth to note here that the Jacobi polynomials is itself a most generalized polynomials and can easily be reduced to or it's limiting cases to many polynomials such as Gegenbauer, Legendre, Hermite, Laguerre, Chebyshev polynomials [37]. Thus from (3.5) to (3.7) one can easily obtain many relations of fractional integrals involving above polynomials.

(v) In (2.1) and (2.3), taking $p = 0; q = 0; p_1 = 2, c_1' = \frac{1}{2}, c_1' = 1;$
$q_1 = 3, d_1' = -n, d_2' = n+1,$
$d_3' = n+1; p_2 = 2, c_1'' = \frac{1}{2}, c_2'' = 1; q_2 = 3, d_1'' = -m, d_2'' = m+1, d_3'' = m+1; \alpha's, \beta's, \gamma's, \delta's$
to be unity in Celine's polynomials, we get fractional integrals of Bessel function of two variables [38].

$$I_{x,y}^{a,b;a',b';c}\left[x^{\alpha+n-a}y^{\beta+m-a'}J_{n,m}(-2\sqrt{x},-2\sqrt{y})\right]$$

$$=\frac{x^{\alpha+n-a-b}\,y^{\beta+m-a'-b'}\,\Gamma(1+\alpha+n-a)\,\Gamma(1+\beta+m-a')}{n!\;m!\quad\Gamma(1+\alpha+n)\;\Gamma(1+\beta+m)}$$

$$\times F_{2:1;1;0;0}^{1:1;1;1;1}\left(\begin{array}{l}(-c:0,0,1,1)\qquad\qquad\qquad\qquad\qquad\quad:(1+\alpha+n-a,1);(1+\beta+m-a',1);\\(1+\alpha+n:1,0,1,0),(1+\beta+m:0,1,0,1):(n+1,1);(m+1,1)\qquad\qquad\qquad\quad;\end{array}\right.$$

$$\left.\begin{array}{l}(a+b,1);(a'+b',1);\\\quad -;\qquad\qquad -;\end{array}x,y,1,1\right),\qquad(3.9).$$

$$J_{x,y}^{a,b;a',b';c}\left[x^{a+b-\alpha-n}y^{a'+b'-\beta-m}J_{n,m}\left(-\frac{2}{\sqrt{x}},-\frac{2}{\sqrt{y}}\right)\right]$$

$$=\frac{x^{a-\alpha-n}\,y^{a'-\beta-m}\,\Gamma(\alpha+n-a)\,\Gamma(\beta+m-a')}{n!\;m!\quad\Gamma(\alpha+n)\;\Gamma(\beta+m)}$$

$$\times F_{2:1;1;0;0}^{1:1;1;1;1}\left(\begin{array}{l}(-c:0,0,1,1)\qquad\qquad\qquad\qquad\quad:(\alpha+n-a,1);(\beta+m-a',1);\\(\alpha+n:1,0,1,0),(\beta+m:0,1,0,1):(n+1,1);(m+1,1)\qquad\qquad\qquad\quad;\end{array}\right.$$

$$\left.\begin{array}{l}(a+b,1);(a'+b',1);\\\quad -;\qquad\qquad -;\end{array}\frac{1}{x},\frac{1}{y},1,1\right),\qquad(3.10).$$

(vi) In (2.1) and (2.3) specializing parameters of fractional operator and Celine's polynomials, we get the fractional integral (Saigo's fractional integral) of Celine's polynomials of single variables as follows

$$I_{x}^{a,b,c}\left[x^{\alpha+n-a}f_{n}(x)\right]=\frac{x^{\alpha+n-a-b}\,\Gamma(1+\alpha+n-a)}{\Gamma(1+\alpha+n)}$$

$$\times F_{1:q+2;0}^{0:p+3;2}\left(\begin{array}{l}\qquad -:-n,n+1,1+\alpha+n-a,a_{1},\ldots,a_{p};-c,a+b;\\1+\alpha+n:\tfrac{1}{2},1,b_{1},\ldots,b_{q}\qquad\qquad\qquad\qquad;\quad -;\end{array}x,1\right),\quad(3.11).$$

$$J_{x}^{a,b,c}\left[x^{a+b-\alpha-n}f_{n}\left(\tfrac{1}{x}\right)\right]=\frac{x^{a-\alpha-n}\,\Gamma(\alpha+n-a)}{\Gamma(\alpha+n)}$$

$$\times F^{0:p+3:2}_{1:q+2;0}\left(\begin{array}{l} -:-n,n+1,\alpha+n-a,a_1,\ldots,a_p;-c,a+b;\\ \alpha+n:\frac{1}{2},1,b_1,\ldots,b_q\qquad\qquad\quad ;\qquad\quad -; \end{array}\frac{1}{x},1\right), \quad (3.12).$$

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FIXED POINT THEOREMS IN METRIC SPACES FOR WEAK** COMMUTING MAPPINGS

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ABSTRACT. Aim of the present paper is to establish some common fixed point theorems for weak** commuting mappings which generalise several previous results.

Keywords. Fixed point, Weak** commuting mappings, Rotative mappings, Metric spaces.

AMS Subject Classification : 47 H 10, 54 H 25

INTRODUCTION

The concept of fixed point theorems with a rational inequality on a weak commutativity condition of mappings have been brought out by Diviccaro,Sessa and Fisher [1], Sessa [5] and Fisher [2]. In this paper we extended the above theory for three self maps of a complete metric space by using the concepts of weak** commuting maps and rotativity of maps [3,4].

RESULTS

Theorem 1. Let *S, I* and *T* be three mappings of a complete metric space (X, *d*) such that for all *x, y* in X

(1) $d(S^2x,\ T^2y) \leq k'd\ (I^2y,\ T^2y)$

$$+\frac{K[d(I^2x,\ S^2x)d(I^2y,\ T^2y)+d(I^2x,\ T^2y)d(I^2y,\ S^2x)]}{d(I^2x,\ S^2x)+d(I^2y,\ T^2y)}$$

$$+\frac{K''[d(I^2x,\ S^2x)d(I^2y,\ S^2x)+d(I^2y,\ T^2y)d(I^2x,\ T^2y)]}{d(I^2x,\ T^2y)+d(I^2y,\ S^2x)}$$

if $d(I^2x,S^2x)+d(I^2y,T^2y) \neq 0$ and $d(I^2x,T^2y)+d(I^2y,S^2x) \neq 0$ where $0<k<1$, $0<k''<1$ and $k + 2k' + 2k'' < 2$ where k' is non negative real and (2) $d(S^2x,T^2y) = 0$ if $d(I^2x,S^2x)+d(I^2y,T^2y) = 0$ or
if $d(I^2x,T^2y)+d(I^2y, S^2x) = 0.$
Suppose that the range of I^2 contains the range of S^2 and T^2 and if either

(3) I^2 is continuous, I is weak** commuting with S, T is rotative w.r.t. I
or
(4) I^2 is continuous, I is weak** commuting with T, S is rotative w.r.t. I
or
(5) S^2 is continuous, S is weak** commuting with I, T is rotative w.r.t. S
or
(6) T^2 is continuous, T is weak** commuting with I, S is rotative w.r.t. T.

Then S,T and I have a common fixed point z. Further, z is the unique common fixed point of S, I and T, I. Also it is a unique common fixed point for I,S and T.

Proof . As the range of I^2 contains the range of S^2, for any arbitrary point x_0 in X, there exists a point x_1 in X such that $S^2x_0 = I^2x_1$. Since the range of I^2 contains the range of T^2, we can choose a point x_2 such that $T^2x_1 = I^2x_2$ .In general, we have

$$S^2x_{2n}=I^2x_{2n+1} \text{ and } T^2x_{2n+1}= I^2x_{2n+2}, \text{ for } n = 0,1,2, \ldots \text{ Now taking}$$

$$d_{2n-1} = d(T^2x_{2n-1} , S^2x_{2n}) \text{ and } d_{2n}=d(S^2x_{2n}, T^2x_{2n+1}) \text{ for } n = 1, 2, \ldots$$

Now we identify three cases.

Case I. $d_{2n-1} \neq 0$ and $d_{2n} \neq 0$ for n = 1, 2, ... Then using inequality (1), we have

$$d_{2n}= d(S^2x_{2n}, T^2x_{2n+1}) \leq k' d(I^2x_{2n+1} ,T^2x_{2n+1})$$

$$+\frac{K[d(I^2x_{2n}, S^2x_{2n})d(I^2x_{2n+1}, T^2x_{2n+1})+d(I^2x_{2n}, T^2x_{2n+1})d(I^2x_{2n+1}, S^2x_{2n})}{d(I^2x_{2n}, S^2x_{2n})+d(I^2x_{2n+1}, T^2x_{2n+1})}$$

$$+\frac{K''[d(I^2x_{2n}, S^2x_{2n})d(I^2x_{2n+1}, S^2x_{2n})+d(I^2x_{2n+1}, T^2x_{2n+1})d(I^2x_{2n}, T^2x_{2n+1})]}{d(I^2x_{2n}, T^2x_{2n+1})+d(I^2x_{2n+1}, S^2x_{2n})}$$

i.e. $d_{2n} \leq k' d_{2n} + k\, d_{2n-1}d_{2n}/(d_{2n-1}+d_{2n}) + k'' d_{2n}$

or $(1-k'-k'')(d_{2n-1}+ d_{2n}) \leq k\, d_{2n-1}.$

$(1-k'-k'')d_{2n} \leq d_{2n-1}(k+k'+k''-1)$

i.e. $d_{2n} \leq [(k+k'+k''-1)/(1-k'-k'')]d_{2n-1}$

which implies $d_{2n} < d_{2n-1}$, since $k+2k'+2k''<2$. Then

(7) $d(S^2x_{2n}, T^2x_{2n-1}) \leq d(T^2x_{2n-1}, S^2x_{2n})$ for n = 1, 2, ...

Now

$$d_{2n-1} = d(T^2x_{2n-1}, S^2x_{2n}) = d(S^2x_{2n}, T^2x_{2n-1})$$

i.e. $d(S^2x_{2n}, T^2x_{2n-1}) \leq k'd(I^2x_{2n-1}, T^2x_{2n-1})$

$$+\frac{K[d(I^2x_{2n}, S^2x_{2n})d(I^2x_{2n-1}, T^2x_{2n-1})+d(I^2x_{2n}, T^2x_{2n-1})d(I^2x_{2n-1}, S^2x_{2n})]}{d(I^2x_{2n}, S^2x_{2n})+d(I^2x_{2n-1}, T^2x_{2n-1})}$$

$$+\frac{K''[d(I^2x_{2n}, S^2x_{2n})d(I^2x_{2n-1}, S^2x_{2n})+d(I^2x_{2n-1}, T^2x_{2n-1})d(I^2x_{2n}, T^2x_{2n-1})]}{d(I^2x_{2n}, T^2x_{2n-1})+d(I^2x_{2n-1}, S^2x_{2n})}$$

This implies $d_{2n-1} \leq k'd_{2n-2} + \dfrac{Kd_{2n-1}d_{2n-2}}{d_{2n-1}+d_{2n-2}} + K''d_{2n-1}$.

or $(1-k'')\, d^2_{2n-1} + (1-k-k'-k'')\, d_{2n-1}\, d_{2n-2} - k'd^2_{2n-2} \leq 0.$

Solving this equation for d_{2n-1}, we have

$$d_{2n-1} = \frac{(k'' + k' + k - 1) \pm \sqrt{(1+ k' + k'')2 + k(k+2k' + 2k''-2)}\; d_{2n-1}}{2(1-k'')}$$

which implies, on taking the positive root,

$$d_{2n-1} \leq [(k''+k'+k-1+1+k'-k'')/2(1-k'')]d_{2n-2}.$$

Since $2k'+2k''+k<2$ and $k > 0$ i.e. $d_{2n-1} \leq [(2k'+k)/2(1-k'')]d_{2n-2}$ which further implies that $d_{2n-1} \leq d_{2n-2}$ and $2k'+k+2k''<2$.
The result is similar when we consider the negative root also. Similarly, in this way we can show that $d_{2n-2} < d_{2n-3}$. Thus
$d(T^2x_{2n-1}, S^2x_{2n}) < d(S^2x_{2n-2}, T^2x_{2n-1})$ for n = 1,2,. . .
It follows that the sequence
(8) $S^2x_0, T^2x_1, S^2x_2, \ldots T^2x_{2n-1}, S^2x_{2n}, T^2x_{2n+1}, \ldots$
is a Cauchy sequence in X. By completeness of X this sequence and its subsequences $\{S^2x_{2n}\}=\{I^2x_{2n+1}\}$ and $\{T^2x_{2n-1}\}=\{I^2x_{2n}\}$ converge to the point w in X.

First let us take I^2 to be continuous, then the sequences $\{I^4x_{2n}\}$ and $\{I^2S^2x_{2n}\}$ converge to the point I^2w. If I weak** commutes with S, we have

$$d(S^2I^2x_{2n}, I^2w) \leq d(S^2I^2x_{2n}, I^2S^2x_{2n}) + d(I^2S^2x_{2n}, I^2w)$$

$$\leq d(S^2x_{2n}, I^2x_{2n}) + d(I^2S^2x_{2n}, I^2w).$$

Letting n→ ∞ we have the sequence $\{S^2I^2x_{2n}\}$ also converges to I^2w .

We now claim that $T^2w = I^2w$. Suppose if not, then we have $d(I^2w,T^2w) > 0$ and using the inequality (1) we obtain,

$$d(S^2I^2x_{2n},T^2w) \leq k'd(I^2w,T^2w)$$

$$+ \frac{K[d(I^4x_{2n}, S^2I^2x_{2n})d(I^2w, T^2w)+d(I^4x_{2n}, T^2w)d(I^2w, S^2I^2x_{2n})]}{d(I^2x_{2n}, S^2I^2x_{2n})+d(I^2w, T^2w)}$$

$$+ \frac{K''[d(I^4x_{2n}, S^2I^2x_{2n})d(I^2w, S^2I^2x_{2n})+d(I^2w, T^2w)d(I^4x_{2n}, T^2w)]}{d(I^4x_{2n}, T^2w)+d(I^2w, S^2I^2x_{2n})}$$

Letting n→ ∞ on both the sides gives

$$d(I^2 w, T^2 w) \leq k'd(I^2 w, T^2 w) + k''d(I^2 w, T^2 w).$$

i.e. $(1- k' - k'')\, d(I^2 w, T^2 w) \leq 0$
which is a contradiction since $2k' + 2k'' + k < 2$ and $k > 0$
Now suppose that $S^2w \neq T^2w$, then

$$d(S^2w,T^2w) \leq k'd(I^2w,T^2w)$$

$$+ \frac{K[d(I^2w, S^2w)d(I^2w, T^2w)+d(I^2w, T^2w)d(I^2w, S^2w)}{d(I^2w, S^2w)+d(I^2w, T^2w)}$$

$$+ \frac{K''[d(I^2w, S^2w)d(I^2w, S^2w)+d(I^2w, T^2w)d(I^2w, T^2w)]}{d(I^2w, T^2w)+d(I^2w, S^2w)}$$

i.e. $d(S^2w, T^2w) \leq k''d(I^2w, S^2w)$

or $(1-k'')d(S^2w, T^2w) \leq 0$, a contradiction as $k''<1$.

Thus we arrive at the result $I^2w = S^2w = T^2w$. A similar conclusion can be obtained if I weak** commutes with T. Let us now suppose that S^2 is continuous instead of I^2. Then the sub sequences $\{S^4x_{2n}\}$ and $\{S^2I^2x_{2n}\}$ converge to the point S^2w. Since S weak** commutes with I, we see that the sub sequence $\{I^2S^2x_{2n}\}$ also converges to S^2w. Since the range of I^2 contains the range of S^2, there exists a point w' such that $I^2w' = S^2w$. If $T^2w' \neq S^2w=I^2w'$, then we have,

$$d(S^4x_{2n}, T^2w') \leq K'd(I^2w', T^2w')$$

$$+ \frac{K[d(I^2x_{2n}, S^4x_{2n})d(I^2w', T^2w')+d(I^2S^2x_{2n}, T^2w')d(I^2w', S^4x_{2n})]}{d(I^2S^2x_{2n}, S^4x_{2n})+d(I^2w, T^2w')}$$

$$+ \frac{K''[d(I^2S^2x_{2n}, S^4x_{2n})d(I^2w', S^4x_{2n})+d(I^2w', T^2w')d(I^2S^2x_{2n}, T^2w')]}{d(I^2S^2x_{2n}, T^2w')+d(I^2w', S^4x_{2n})}$$

and on letting n→ ∞, we get

$$d(S^2w, T^2w') \leq k'd(I^2w', T^2w')$$

$$+ \frac{K[d(S^2w, S^2w)d(I^2w', T^2w')+d(S^2w, T^2w)d(I^2w', S^2w)]}{d(S^2w, S^2w)+d(I^2w', T^2w')}$$

$$+ \frac{K''[d(S^2w, S^2w)d(I^2w', S^2w)+d(I^2w', T^2w')d(S^2w, T^2w')]}{d(S^2w, T^2w')+d(I^2w', S^2w)}$$

Using $S^2w = I^2w'$, we get

$$d(S^2w, T^2w') \leq k'd(I^2w', T^2w')+k''d(I^2w', T^2w').$$

i.e. $(1-k'-k'')d(S^2w, T^2w') \leq 0$ which is a contradiction as $k'+k'' < 1$. Thus we arrive at the result $S^2w=T^2w'=I^2w'$.

Suppose $S^2w' \neq T^2w'=I^2w'$ then,

$$\begin{aligned} d(S^2w',T^2w') \le\ & k'(I^2w',T^2w') \\ & + \frac{k\,[d(I^2w',S^2w')\,d(I^2w',T^2w') + d(I^2w',T^2w')d(I^2w',S^2w')]}{d(I^2w',S^2w') + d(I^2w',T^2w')} \\ & + \frac{k''\,[d(I^2w',S^2w')\,d(I^2w',S^2w') + d(I^2w',T^2w')d(I^2w',T^2w')]}{d(I^2w',T^2w') + d(I^2w',S^2w')} \end{aligned}$$

i.e. $d(S^2w',T^2w') \le k''d(S^2w',T^2w')$

or $(1-k'')\,d\,(S^2w',T^2w') \le 0$

which is again a contradiction as $k'' < 1$. Thus $I^2w'=S^2w'=T^2w'$. If T^2 is continuous and T is weak** commuting with I, we can arrive at the same result .

Case II. Let $d_{2n-1} = 0$ for some n. Then $I^2x_{2n} = T^2x_{2n-1} = S^2x_{2n}$

We claim that $I^2x_{2n} = T^2x_{2n}$. We suppose not, then $d\,(I^2x_{2n}, T^2x_{2n}) > 0$ then inequality (1) implies,

$$\begin{aligned} 0 < d(I^2x_{2n},T^2x_{2n}) &= d(S^2x_{2n}, T^2x_{2n}) \\ &\le k'd(I^2x_{2n}\,,\ S^2x_{2n}\ ) \\ &\quad + \frac{K[d(I^2x_{2n}, S^2x_{2n})d(I^2x_{2n}, T^2x_{2n}) + d(I^2x_{2n}, S^2x_{2n})d(I^2x_{2n}, S^2x_{2n})]}{d(I^2x_{2n}, S^2x_{2n}) + d(I^2x_{2n}, T^2x_{2n})} \\ &\quad + \frac{K''[d(I^2x_{2n}, S^2x_{2n})d(I^2x_{2n}, S^2x_{2n}) + d(I^2x_{2n}, T^2x_{2n})d(I^2x_{2n}, T^2x_{2n})]}{d(I^2x_{2n}, T^2x_{2n}) + d(I^2x_{2n}, T^2x_{2n})} \end{aligned}$$

$$0 < d\,(\,I^2x_{2n}\,,\ T^2x_2n\,) \le k''d\,(\,I^2x_{2n}\,,\ T^2x_{2n}\,).$$

i.e. $$0 < (1-k'')d(I^2x_{2n}\,,\ T^2x_{2n}) \le 0$$

which is a contradiction as $k'' < 1$. Thus $I^2x_{2n} = S^2x_{2n} = T^2x_{2n}$

Case-III. Let $d_{2n} = 0$ for some n, then $I^2x_{2n+1} = S^2x_{2n} = T^2x_{2n+1}$, with the similar proof as in case-II we have,

$$I^2x_{2n+1} = S^2x_{2n+1} = T^2x_{2n+1}.$$

Therefore in all the three cases, there exists a point w such that $I^2w = S^2w = T^2w = z$ (say). If I weak** commutes with S, we have

$$d(S^2Iw, IS^2w) \le d(SI^2w, I^2Sw) \le d(SIw, ISw) \le d(S^2w, I^2w) = 0$$

which implies that

(9) $S^2Iw = IS^2w$, $SI^2w = I^2Sw$,

$SIw = ISw$, and so $I^2Sw = S^3w$.

If x=Sw and y=w then by (9)

$d(I^2x, S^2x) + d(I^2y, T^2y) = o$ and so (2) implies $Sz = z$.

It follows that $I^2w = z$ is a fixed point of S. Further if x=Iw and y=w then again by (9),

$$d(I^2x, S^2x) + d(I^2y, T^2y) = 0$$

and so (2) implies Iz=z.

Now we claim that $T^2z = z$. If not we have,

$$d(z, T^2z) = d(S^2z, T^2z) \le k'd(I^2z, T^2z)$$

$$+ \frac{K[d(I^2z, S^2z)d(I^2z, T^2z) + d(I^2z, T^2z)d(I^2z, S^2z)]}{d(I^2z, S^2z) + d(I^2z, T^2z)}$$

$$+ \frac{K''[d(I^2z, S^2z)d(I^2z, S^2z) + d(I^2z, T^2z)d(I^2z, T^2z)]}{d(I^2z, T^2z) + d(I^2z, S^2z)}$$

i.e. $d(z, T^2z) \le k'd(z, T^2z) + K''d(z, T^2z)$

or $(1 - k' - k'')\, d(z, T^2z) \le 0$ which is a contradiction as $k' + k'' < 1$ and so $T^2z = z$. Now using the rotativity of T w.r.t I we have

$$d(Tz, z) = d(Tz, I^2z) \le d(Iz, T^2z) = d(z, z) = 0$$

And so, z is a common fixed point of T.

The proof is similar in other cases (4), (5) and (6).

Now to prove the uniqueness of the point z, let z' be another common fixed point of I and S. Then $d(I^2z, S^2z') + d(I^2z, T^2z) = 0$ and the condition (2) implies that,

$$z' = Sz' = S^2z' = T^2z = z.$$

In the same way we can show that z is a unique common fixed point of I and T. This completes the proof of the theorem.

Example 1. Let X be the subset of R^2 defined by $X = \{A,B,C,D,E\}$ where $A \equiv (0,0)$, $B \equiv (1,0)$, $C \equiv (0,1)$, $D \equiv (1/2,0)$, $E \equiv (-1,0)$
Let $I, S, T : X \to X$ be given by
$I(A) = I(B) = I(C) = B$, $I(D) = A$, $I(E) = D$,
$S(A) = S(B) = S(C) = B$, $S(D) = S(E) = A$,
$T(A) = T(B) = T(C) = T(D) = T(E) = B$.

By this assumption, it is obvious that I weak** commutes with S, T is rotative w.r.t I. Clearly I^2 (or S^2) is continuous, $S^2(X)=\{B\} \subseteq \{A,B\}=I^2(X)$ and $T^2(X) = \{B\} \subseteq \{A , B) = I^2(X)$. We shall verify whether the conditions of the Theorem 1 are satisfied here. In the condition (1) taking $x = A$, $y = E$, we have

$$d(S^2A,T^2E) \le k'd(I^2E,T^2E)$$

$$+ \frac{K[d(I^2A, S^2A)d(I^2E, T^2E)+d(I^2A, T^2E)d(I^2E, S^2A)]}{d(I^2A, T^2A)+d(I^2E, T^2E)}$$

$$+ \frac{K''[d(I^2A, S^2A)d(I^2E, S^2A)+d(I^2E, T^2E)d(I^2A, T^2E)]}{d(I^2A, T^2E)+d(I^2E, S^2A)}$$

i.e. $$d(B,B) \le k'd(A,B) + \frac{K[d(B, B)d(A, B)+d(B, B)d(A, B)]}{d(B, B)+d(A, B)}$$

$$+ \frac{k''[d(B,B)\,d(A,B) + d(A,B)\,d(B,B)]}{d(B,B)\,d(A,B)}$$

and so $0 \le k'\,d(A,B) + 0 + 0$ which is true. Thus with $k+ 2k'+2k'' < 2$ and $k > 0$, the condition (2) also holds for the points $x, y \in \{A,B,C,D\}$. Therefore both the conditions of the theorem 1.2 are satisfied and hence B is the unique common fixed point of I,S and T.

Corollary 2. Let S and I be mappings of a complete metric space (X,d) into itself, such that for $x,y \in X$ either

$$d(S^2x, S^2y) \leq k'd(I^2y, S^2y)$$

$$+\frac{K[d(I^2x, S^2x)d(I^2y, S^2y)+d(I^2x, S^2y)d(I^2y, S^2x)]}{d(I^2x, S^2x)+d(I^2y, S^2y)}$$

$$+\frac{K''[d(I^2x, S^2x)d(I^2y, S^2x)+d(I^2y, S^2y)d(I^2x, S^2y)]}{d(I^2x, S^2y)+d(I^2y, S^2x)}$$

where $d(I^2x, S^2x)+d(I^2y,S^2y) \neq 0$ and $d(I^2x,S^2y)+d(I^2y,S^2x) \neq 0$

where $k+2k'+2k'' < 2$ and $k>0$

or $d(S^2x,S^2y) = 0$ if $d(I^2x,S^2x)+d(I^2y,S^2y) = 0$

or $d(I^2x, S^2y)+d(I^2y, S^2x)=0$.

If I and S are weak ** commuting mappings, $S^2(X) \subseteq I^2(X)$ and S^2 or I^2 is continuous, then S and I have a unique common fixed point.

We shall now see in an example which reconfirms that the two conditions of the above corollary are essential for the existence of unique fixed point.

Example 2. Let $X = \{x,y\}$ with the discrete metric. I and S are defined on X by,

$$Ix = Iy = x, Sx = Sy = y.$$

Since X is discrete, $d(S^2x,S^2y) = 0$ for all x, y in X, all the conditions of the corollary 2 are satisfied except that $S^2(X) \subseteq I^2(X)$, but S and I have no common fixed points.

Corollary 3

Taking $I=I^2x$, the identity map on X and dropping the rotativity of T (or S) we have the corollary -

Let S and T be mappings of a complete metric space (X, d) into itself such that for all x, y in X.

$$d(S^2x, T^2y) \leq k'd(y, T^2y) + \frac{K[d(x, S^2x)d(y, T^2y)+d(x, T^2y)d(y, S^2x)]}{d(x, S^2x)+d(y, T^2y)}$$

$$+ \frac{K''[d(x, S^2x)d(y, S^2x)+d(y, T^2y)d(x, T^2y)]}{d(x, T^2y)+d(y, S^2x)}$$

if $d(x,S^2x)+d(y,T^2y) \neq 0$ and $d(x, T^2y)+d(y, S^2x) \neq 0$

where $k+2k'+2k'' < 2$ and $k > 0$ or

(10) $d(S^2x,T^2y)=0$ if $d(x,S^2x)+d(y,T^2y)=0$ or $d(x,T^2y)+d(y,S^2x)=0$.

Then S and T have a unique common fixed point z. Further z is the unique fixed point of S and T .

Proof. It is obvious that there exists a point $w \in X$ such that, $w = S^2w = T^2w$.

Thus $d(Sw,S^2Sw)+d(w,T^2w)=0$ and using condition (10) we see that $Sw=S^2Sw=T^2w$.

Further $d(w,S^2w)+d(Tw,T^2Tw)=0$ and using condition (10), we deduce that $Tw = T^2Tw = Sw = w$.

It follows that w is a common fixed point of S and T. The uniqueness of w can be proved in the usual way. This completes the proof .

REMARKS

(i) It is clear from the proof of Theorem 1 that if condition (2) is omitted in the statement, we can prove that w is a coincidence point of I^2, S^2 and T^2.

(ii) Assuming that I,S and T are idempotent maps of X and K'=0=K", we obtain theorem 1 of [1].

(iii) Assuming I as identity map and S and T as idempotent maps of X and K' = 0 = K", we obtain theorem 3 of [2].

(iv) Assuming I , S and T as idempotent maps of X, and S=T on X, and K' = 0 = K", we obtain corollary 2 of [1].

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ALMOST AUTOMORPHIC FUNCTIONS IN NON-ARCHIMEDEAN BANACH SPACE

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ABSTRACT. The aim of the present paper is to make the study of a.a functions in a non Archimedean Banach space and studied some of its Properties.

Keywords. Banach space, Non-Archimedean Banach space, Automorphic function.

1. INTRODUCTION

Almost automorphic(a.a) functions are related class of almost periodic functions. These functions were first introduced by Bochner [1] in a paper from R into R. Later on these functions were studied by many researchers. We specifically considered Veech [3] and Zaki [5].Fink [2] discussed the existence of (a.a) solutions. Zaidman [4] studied (a.a) functions from real line into Banach space. The aim of this paper is to make the study of (a.a) functions in a non Archimedean Banach space.

Simple Properties of a.a Functions.

Let x be a non-Archimedean Banach space then a function $f(t) : R \rightarrow X$ is called a.a if from any real sequence {a¢n} we can extract a subsequence $\{\alpha n\}$such that
$\lim_{n \rightarrow \infty} f(t + \alpha_n) = g(t), t \in R$ and $\lim_{n \rightarrow \infty} g(t - \alpha_n) = f(t) \; \forall \quad t \in R$.

It is true, that above limits hold in case of almost periodic function, but the mode of convergence is uniform over the real line R. For a.a function we require only pointwise convergence. Consequently, the function g(t) need not be continuous but defined for all $t \in R$.

The sum and difference of two a.a functions are almost automorphic. If an almost automorphic function is multiplied by a scalar then result is also a.a. All constant functions are a.a.

Proposition 1. Every almost automorphic function is bounded with respect to non-Archimedean norm over R.

Proof. If possible there exists a real sequence $\{t_n\}$ such that $\| f(t_n) \| \geq n$, n= 1, 2, ... but by definition we can extract a subsequence $\{t_{n_i}\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} f\{t_{n_i}\} = g(0)$ exists. This would imply boundedness of the sequence $\{f(t_{n_i})\}$. Which is a contradiction.

Proposition 2. If f (t) is a.a then reflected function $\check{f}(t) = f(-t)$ is also a.a.

Proof. Let $\{\alpha'_n\}$ be any real sequence then $\{-\alpha'_n\}$ is opposite sequence, and from this opposite sequence we extract a subsequence $\{-\alpha_n\}$ such that $\lim_{n\to\infty} f\{t-\alpha_n\}=g(t)$ also $\lim_{n\to\infty} g\{t+\alpha_n\}= f(t) \quad \forall \quad t \in R$

therefore it follows that

$$\lim_{n\to\infty} \check{f}\{t+\alpha_n\} = \lim_{n\to\infty} f\{-t-\alpha_n\} = g(-t) = \check{g}(t) \quad \text{and}$$

$$\lim_{n\to\infty} \check{g}\{t-\alpha_n\} = \lim_{n\to\infty} g\{-t+\alpha_n\} = f(-t) = \check{f}(t)$$

so that f (t) is almost automorphic.

The intermediate function g(t) is also bounded over R. Infact, rang $R_g \subseteq R_f$ is compact in x, also $\sup \|g(t)\| \leq \sup \|f(t)\|$. The range R_t of any a.a function is precompact.

2. REMARK

In order to ease the notation for taking limits Bochner introduced the translation operator T. $T_\alpha f = g$ means that $g(t) = \lim_{n\to\infty} f(t+\alpha_n)$ and is written only when the limit exists pointwise, If the sequence $\{\alpha_n\}$ is denoted by α and $\alpha \subset \alpha'$ means that the sequence $\{\alpha_n\}$ is a subsequence of $\{\alpha'_n\}$ then the definition a.a function becomes : f is a.a is for every α' there is $\alpha \subset \alpha'$ such that $T\alpha$ f exists pointwise.

3. MAIN RESULT

Theorem. The a.a is preserved under uniform convergence on the real line (See Veech [3]).

Proof. We consider a sequence $\{f_j(t)\}_{j=1}^{\infty}$ of a.a function in a non-Archimedean Banach space defined on real line R and we assume that $f_j(t)$ tends to limit f(t) uniformly on R. Then f(t) is also a.a from $R \rightarrow X$. Let $\{\alpha'_n\}$ be any arbitrary real sequence we extract a subsequence $\{\alpha n\}$ such that

$$f_j(t + \alpha_n) \rightarrow g_j(t) \quad \text{and} \quad g_j(t - \alpha_n) \rightarrow f_j(t), \qquad t \in R,\ j = 1, 2, 3, \ldots$$

But the sequence $\{g_j(t)\}_{j=1}^{\infty}$ of bounded functions from R into X is a cauchy sequence with respect to the uniform convergence on R. Infact, for any $t \varepsilon R$,

$$\|g_j(t) - g_k(t)\| = \|g_j(t) - f_j(t + \alpha_n) + f_j(t + \alpha_n) - f_k(t + \alpha_n) + f_k(t + \alpha_n) - g_k(t)\|$$

$$\leq \max\{\|g_j(t) - f_j(t + \alpha_n)\|, \|f_j(t + \alpha_n) - f_k(t + \alpha_n)\|, \|f_k(t + \alpha_n) - g_k(t)\|\}$$

$$\leq \max\{\varepsilon, \|g_j(t) - f_j(t + \alpha_n)\|, \|f_k(t + \alpha_n) - g_k(t)\|\},$$

where $j, k \geq N(\varepsilon)$ (by uniform convergence of $f_j(t)$ to f(t).
As $n \rightarrow \infty$, in both sides (right and left), we obtain

$$\|g_j(t) - g_k(t)\| < \varepsilon, \qquad j, k \geq N(\varepsilon)$$

where t is any real number and N is an independent of t. Therefore, we have the uniform convergence of the sequence $\{g_j(t)\}$ over the real line. let $g(t) : R \rightarrow X$ be the uniform limit of sequence. We have $\forall\, t \in R$, $f(t + \alpha_n) \rightarrow g(t)$ and $g(t - \alpha_n) \rightarrow f(t)$. We have to prove for the second limit as,

$$\|g(t - \alpha_n) - f(t)\| = \|g(t - \alpha_n) - g_j(t - \alpha_n) + g_j(t - \alpha_n) - f_j(t) + f_j(t) - f(t)\|$$

$$\le \max\{\|g(t-\alpha_n)-g_j(t-\alpha_n)\|, \|g_j(t-\alpha_n)-f_j(t)\| \|f_j(t)-f(t)\|\};$$

For any given $\varepsilon > 0$, we have $\|g(t-\alpha_n)-g_j(t-\alpha_n)\| < \varepsilon/2$

and $\|f_j(t)-f(t)\| < \varepsilon/2$ where $j \ge J(\varepsilon)$.

Thus, we get $\|g(t-\alpha_n)-f(t)\| \le \max\{\varepsilon/2, \varepsilon/2, \|g_j(t-\alpha_n)-f_j(t)\|\}$

we have $N(t, J_\varepsilon)$ such that $\|g_j(t-\alpha_n)-f_j(t)\| \le \varepsilon/2, \quad n \ge N$

Consequently $\|g(t-\alpha_n)-f(t)\| \le \max\{\varepsilon/2, \varepsilon/2, \varepsilon/2,\}$

or $\|g(t-\alpha_n)-f(t)\| \le \varepsilon$ for $n \ge N_1(t, \varepsilon)$

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COMMON FIXED POINT THEOREMS FOR SIX MAPPINGS IN FUZZY METRIC SPACE

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ABSTRACT. In this note we prove some common fixed point theorems for six mappings on fuzzy metric space using notion of compatibility and taking the continuously of maps.

Key words. Fuzzy metric space, Compatible mappings, Common fixed point.
AMS subject classification, Primary 54H25, Secondary 47H10.

1. INTRODUCTION

The concept of fuzzy set was introduced by Zadeh [6] in his classic paper. This theory evolved in many directions and has applications in wide variety of fields in which the phenomenon under the studies are too complex. To use this concept in topology and analysis, several researchers' defined fuzzy metric space in various ways.

Singh and Chauhan [5] introduced a lighter concept of compatibility than that of weak commutativity of maps. The concept of compatibility was first introduced by Jungck [4]. Result of Imadad and Khan [2], Jeong and Rhoades [3] are also mentioned.

In this paper, using the concept of compatibility , we generalize the result of Singh and Chauhan [5] for six mappings as opposed to four in complete fuzzy metric spaces [1].

2. RESULT

For the terminologies and basic properties of fuzzy metric spaces the reader is referred to George and Veermani [1], Singh and Chauhan [5].

Theorem 2.1 Let A,B,S,T,I and J be self mappings of a complete fuzzy metric space ($X,M,*$) with continuous t-norm defined by $a * b = \min\{a,b\}$, $a,b \in [0,1]$ satisfying the following conditions

$AB(X) \subset J(X)$, $ST(X) \subset I(X)$ for all $x,y \in X$, $t>0$ (2.1.1)

mappings I and J are continuous, (2.1.2)

$\{AB,I\}$, $\{ST, J\}$ are compatible pairs of mappings. (2.1.3)

for all x,y in X, $t>0$, $k\in(0,1)$, (2.1.4)

$$M(ABx, STy, kt) \geq \min\{M(ABx, Ix, t), M(STy, Jy, t), M(Ix, Jy, 2t),$$
$$M(ABx, Jy, 2t), M(STy, Ix, 2t)\}$$

for all x,y in X, $\lim M(x,y,t) \to 1$ as $t \to \infty$ (2.1.5)

Then AB,ST,I and J have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Since $AB(X) \subset J(X)$ we can find a point x_1 in X such that $ABx_0 = Jx_1$. Also since $ST(X) \subset I(X)$ we can choose a point x_2 with $STx_1 = Ix_2$. Using this argument repeatedly one can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$x_{2n-1} = Jx_{2n-1} = ABx_{2n-2} \text{ and } y_{2n} = Ix_{2n} = STx_{2n-1} : n=1, 2, \ldots$$

From (2.1.4) we have

$$M(y_{2n+1}, y_{2n+2}, kt) = M(ABx_{2n}, STx_{2n+1}, kt)$$

$$\geq \min\{M(ABx_{2n+2}, Ix_{2n+2}, t), M(STx_{2n+1}, Jx_{2n+1}, t)$$

$$M(Ix_{2n+2}, Jx_{2n+1}, t), M(ABx_{2n+2}, Jx_{2n+1}, t), M(STx_{2n+1}, Ix_{2n+2}, t)\}$$

$$\geq \min\{M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t), 1)\}$$

which implies $M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t)$.
In general

$$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t) \qquad (2.1.6)$$

To prove that $\{y_n\}$ is a Cauchy sequence, we shall prove

$M(y_n, y_{n+m}, t) > 1-\lambda$ is true for all $n \geq n_0$ and for every $m \in N$. (2.1.7)

Here we use induction method. From(2.1.6) we have

$$M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, t/k) \geq M(y_{n-2}, y_{n-1}, t/k^2) \geq \ldots M(y_0, y_1, t/k^n) \to 1 \text{ as } n \to \infty$$

i.e. for $t>0$, $\lambda \in (0,1)$ we can choose $n_0 \in N$ such that $M(y_n, y_{n+1}, t) > 1-\lambda$.

Thus (2.1.7) is true for m=1. Suppose (2.1.7) is true for m then we shall prove that it is also true for m+1. Using the definition of fuzzy metric space ,(2.1.6) and (2.1.7) we have

$$M(y_n, y_{n+m+1}, t) \geq \min\{M(y_n, y_{n+m}, t/2), M(y_{n+m}, y_{n+m+1}, t/2)\} > 1-\lambda.$$

Hence (2.1.7) is true for m+1.Thus $\{y_n\}$ is a Cauchy sequence. By completeness of $(X,M,*)$, $\{y_n\}$ converges to some point z in X.

Thus $\{ABx_{2n}\}$, $\{Ix_{2n}\}$, $\{STx_{2n-1}\}$ and $\{Jx_{2n-1}\}$ also converges to z .Now $ABx_{2n} \to z$ and I is continuous hence $IABx_{2n} \to I_z$
Thus for $t>0$, $\lambda \in (0,1)$, there exists an $n_0 \in N$ such that

$M(IABx_{2n}, I_z, t/2) > 1-\lambda$ for all $n \geq n_0$.

Using (2.1.3), we have $M(ABIx_{2n}, IAB\, x_{2n}, t/2) \to 1$

$M(ABIx_{2n}, I_z, t) \geq \min\{M(ABIx_{2n}, IAB\, x_{2n}, t/2), M(IABx_{2n}, I_z, t/2)\} > 1-\lambda.$
Hence

$$ABIx_{2n} \to I_z \,. \qquad (2.1.8).$$

Similarly

$$STJx_{2n-1} \to J_z \qquad (2.1.9).$$

Using (2.1.4) we have

$$\begin{aligned} M(ABIx_{2n}, STJx_{2n-1}, kt) \geq \min\{&M(I^2 x_{2n} . J^2 x_{2n-1}, t), M(ABIx_{2n}, I^2 x_{2n}, t), \\ &M(STJx_{2n-1}, J^2 x_{2n-1}, t), M(STJx_{2n-1}, I^2 x_{2n}, 2t), \\ &M(ABIx_{2n}, J^2 x_{2n-1}, 2t)\} \end{aligned}$$

Taking limit as n → ∞ and using (2.1.8) and (2.1.9) we get

$M(Iz, Jz, kt) \geq M(Iz, Jz, t)$ which implies that

$$Iz = Jz \qquad (2.1.10).$$

Now

$$\begin{aligned} M(ABz, STJx_{2n-1}, kt) \geq \min\{&M(ABz, Iz, t), M(STJx_{2n-1}, J^2 x_{2n-1}, t), \\ &M(Iz, J^2 x_{2n-1}, t), M(ABz, J^2 x_{2n-1}, 2t), \\ &M(STJx_{2n-1}, Iz, 2t)\} \end{aligned}$$

Taking the limit as n →∞ and using (2.1.8) and (2.1.10) We get

$$ABz = Jz \qquad (2.1.11).$$

Now using (2.1.10) and (2.1.11), we get

$$\begin{aligned} M(ABz, STz, kt) &\geq \min\{M(Iz, Jz, t), M(ABz, Iz, t), M(STz, Jz, t), \\ &\qquad M(STz, Iz, 2t), M(ABz, Jz, 2t)\} \\ &\geq \min\{M(Jz, Iz, t), M(ABz, ABz, t), M(ABz, STz, t), \\ &\qquad M(ABz, STz, 2t), M(ABz, ABz, t)\} \\ &\geq M(ABz, STz, t) \end{aligned}$$

which implies that

$$ABz = STz \qquad (2.1.12).$$

Using (2.1.10) and (2.1.12) we get

$$ABz = STz = Iz = Jz \qquad (2.1.13).$$

Now

$$M(ABx_{2n}, STz, kt) \geq \min \{M(Ix_{2n}, Jz, t), M(ABx_{2n}, Ix_{2n}, t), M(STz, Jz, t), M(STz, Ix_{2n}, 2t), M(ABx_{2n}, Jz, 2t)\}.$$

Taking limit and using (2.1.13) we get

$$z = STz \qquad (2.1.14).$$

Thus z is a common fixed point of AB , ST ,I and J. For the uniqueness, let w be another common fixed point of said maps. Then from (2.1.4) we have

$$M(ABz, STw, kt) \geq \min \{M(Iz, Jw, t), M(ABz, STz, t), M(STw, Jw, t), M(STw, Iw, 2t), M(ABz, Jw, t)\}$$

i.e. $M(z, w, kt) \geq M(z, w, t)$. Hence $z = w$. This completes the proof.

Theorem 2.2. Let A, B, S, I, J and T be self maps of a complete fuzzy metric space $(X, M, *)$ with continuous t-norm defined by $a*b = \min\{a, b\}$, $a, b \in [0,1]$ satisfying the condition (2.1.1) and (2.1.5) and

$$AB(I) = I(AB),\ J(ST) = ST(J), \qquad (2.2.1)$$

$$(AB)^a(X) \subset J^t(X),\ (ST)^b(X) \subset I^s(X), \qquad (2.2.2)$$

$$M((AB)^a x, (ST)^b y, kt) \geq \min\{M(I^s x, J^t y, t), M((AB)^a x, I^s x, t), M((ST)^b y, J^t y, t), M((ST)^b y, I^s x, 2t), M((ST)^b y, I^s x, t)\} \qquad (2.2.3)$$

for all x,y in X .Then AB,ST,I and J have a unique common fixed point in X.

Proof. Since AB and ST commute with I and J so $(AB)^a$ and $(ST)^b$ also commute with I^s and J^t respectively also commutivity implies compatibility ,hence by previous theorem $(AB)^a$,$(ST)^b$,I^s ,J^t have a unique common point say z in X i.e.

$$z = (AB)^a z = (ST)^b z = I^s z = J^t z.$$

Now $ABz = AB((AB)^a z) = (AB)^a (ABz)$ and $ABz = AB(I^s z) = I^s(ABz)$ hence AB(z) is a common fixed point of $(AB)^a$ and I^s. Similarly it can be shown that ST (z) is a common fixed point of $(ST)^b$ and J^t.

Now putting x = AB(z) and y = ST(z) in (2.2.3) we have

$$\begin{aligned}
M((AB)^a z,(ST)^b z, kt) &= M\{ (AB)^a(z), ST^b (STz), kt\} \\
&\geq \min \{M (I^s(ABz),J^t(STz, t))\ M((AB)^a (ABz), I^s ABz),t), \\
&\qquad M((ST)^b(STz), J^t (STz)\, t), M ((ST)^b(STz),I^s (ABz),2t), \\
&\qquad M((AB)^a (ABz) ,J^t(STz),t)\} \\
&= \min \{M(AB(z),ST(z),t),M(AB(z),,AB(z),t). \\
&\qquad M(ST(z),ST(z),t), M(ST(z),AB(z),2t), \\
&\qquad M(AB(z),ST(z),t)\} \\
&\geq M(AB(z),ST(z),t)
\end{aligned}$$

which implies AB(z) = ST(z). Similarly Iz = Jz.

Since z is unique common fixed point of $(AB)^a$, $(ST)^b$, I^s, J^t and AB(z) (= ST(z)), Iz(= Jz) are common fixed points of $(AB)^a$, I^s and $(ST)^b$, J^t respectively. Hence z = AB(z) = ST(z) = Iz = Jz. This comletes the proof.

EXAMPLE. Let (X,M,∗) be a fuzzy metric space with X={0,1}, t-norm ∗ defined by $a*b = \min(a,b)$, $a,b \in [0,1]$ and M is a fuzzy set on $X^2 *(0,\infty)$, defined by $M(x,y,t) = [\exp (x - y /t)]^{-1}$ for all $x,y \in X$, $t > 0$.

Define self maps A,B,S,T,I and J such that

$Ax = 3x/2$, $Bx = x/24$, $Sx = 3x/4$, $Tx = x/6$, $Ix = x/4$, $Jx = x/2$.

Then for $k \in [\,1/4\ ,1]$

$$\begin{aligned}
M(ABx,STy,kt) &= [\exp(x/16 - y/8) /kt]^{-1} \\
&\geq [\exp(x/4 - y/2) /t\,]^{-1} \\
&= M(Ix , Jy , t) \\
&\geq \min \{M(Ix,Iy,t),M(AB(x),Ix,t), MST(y),Jy,t), \\
&\qquad M(STy,Ix,2t), M(AB(x)Jy,t)\}.
\end{aligned}$$

Thus all the conditions of Theorem 2.1 are satisfied and zero is the unique common fixed point of AB, ST, I and J .

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CUMULATIVE SUM CONTROL CHART FOR POISSON VARIABLES UNDER INSPECTION ERRORS

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ABSTRACT. The effect of inspection errors on Cumulative Sum Control Chart (CSCC) for Poisson variables when one is inspecting for the number of non conformities per item is considered and two types of inspection errors are defined. Formulas are derived for the d, θ and Average Run Length (ARL) for situations when these inspection errors are present. An example illustrates the effect of these inspection errors on d, θ and ARL.

KEY WORDS. Poisson Variables, SPRT, Inspection errors, ARL, CSCC.

1. INTRODUCTION

A principal purpose of inspection is to separate product that conforms to the specifications from that which does not confirm the specifications. Fundamentally, an inspection operation may result in two kinds of errors viz. classification of a non conforming unit as conforming unit and classification of a conforming unit as non conforming. The probabilities of these inspection errors, and the incoming fraction non conforming are the principal factors that determine the performance of an inspection operation. CSCC are widely used to monitor the quality of continuous manufacturing processes. Gibra [1] and Vance [5] provide references on research performed on CSCC. Marquardt [4] recommended their use in U.S. industry since the CSCC can detect small changes in the distribution of a quality characteristic and thus maintain tight control over a process. One does not have to be involved in long inspection before one sees inspection error occurring.

The CSCC for Poisson variables is a well known and widely used tool in process control. An item (or group of items) is sampled, inspected, and the number of non conformities in the sample are noted. The literature on CSCC for Poisson variables assumes that inspections are error free, but this is not the case in practice.

The purpose of this paper is to consider the effect of inspection errors on the CSCC for Poisson variables.

2. CSCC FOR POISSON VARIABLES WITHOUT INSPECTION ERROR

The ordinates of points plotted on the standard C chart can often be interpreted as observed values of a Poisson variable. If the mean value of a Poisson variable is μ the probability that a value x will be observed is

$$P(x) = \frac{e^{-\mu}\mu^{x}}{x!}, \quad x = 0, 1 \ldots \quad \mu > 0. \tag{2.1}$$

The hypothesis under test is

$H_0 : \mu = \mu_0$ *and* $H_1 : \mu = \mu_1$ $(\mu_1 > \mu_0)$ has the continuation region of SPRT as,

$$\frac{\ln\left(\frac{\alpha_1}{1-\alpha_0}\right) + m(\mu_1 - \mu_0)}{\ln\left(\frac{\mu_1}{\mu_0}\right)} < \sum_{i=1}^{m} x_i < \frac{\ln\left(\frac{1-\alpha_1}{\alpha_0}\right) + m(\mu_1 - \mu_0)}{\ln\left(\frac{\mu_1}{\mu_0}\right)}. \tag{2.2}$$

Here α_0 = Probability of accepting H_1 when H_0 is true and
α_1 = Probability of accepting H_0 when H_1 is true.

If we plot points (m, X_m) with $X_m = \sum_{i=1}^{m} x_i$, the boundary line between the continuation region and the acceptance region for H_1, has the equation

$$X_m = \frac{\ln\left(\dfrac{1-\alpha_1}{\alpha_0}\right)}{\ln\left(\dfrac{\mu_1}{\mu_0}\right)} + \frac{\mu_1-\mu_0}{\ln\left(\dfrac{\mu_1}{\mu_0}\right)}.m \qquad (2.3)$$

Using the approximation given by Johnson [3]

$$X_m = \frac{\ln\alpha_0}{\ln\left(\dfrac{\mu_1}{\mu_0}\right)} + \frac{\mu_1-\mu_0}{\ln\left(\dfrac{\mu_1}{\mu_0}\right)}m , \qquad (2.4)$$

the control limits for CSCC on Poisson variables are

$$d = (-\ln \alpha_0)/(\mu_1-\mu_0) \qquad (2.5)$$

$$\theta = \tan^{-1}\left(\frac{\mu_1-\mu_0}{\ln (\mu_1/\mu_0)}\right) \qquad (2.6)$$

Following Johnson [3], the approximate formula for ARL for detecting a shift in the parameter μ from μ_0 to μ_1 is given by

$$ARL = \frac{-\ln\alpha_0}{\mu_1\ln\left(\dfrac{\mu_1}{\mu_0}\right)-(\mu_1-\mu_0)} \qquad (2.7)$$

3. CSCC FOR POISSON VARIABLES WITH INSPECTION ERROR

Let r be the probability that a nonconformity is correctly noted by the inspector. We note that r is assumed to be a constant over different values of μ. In one study reported by Harris and Chaney [2] r varies from 0.58 to 0.80 where as μ varies from 0.0025 to 0.16. However, this variability in r also included the variability among different inspectors, since different inspectors were used for different values of μ. Therefore, the assumption of a constant r does not seen to be too seriously violated, especially noting the large spread in values of μ. If μ_r is the average number of false alarms per item and μ is the true average number of non conformities per part and μ' is the average number per part observed by the inspector, then

$$\mu' = r\,\mu+\mu_r \tag{3.1}$$

with both r and μ_r estimated. Every effort should be made to estimate both types of errors, i.e. to get r close to one and μ_r close to zero. If μ is the target value, then the control limit and ARL under inspection error for CSCC on Poisson variable is obtained by (2.5),(2.6) and (2.7) as follows -

$$d = \frac{(-\ln \alpha_0)}{(\mu'_1 - \mu'_0)} \tag{3.2}$$

$$\theta = \tan^{-1}\left(\frac{(\mu'_1 - \mu'_0)}{\ln (\mu'_1/\mu'_0}\right) \tag{3.3}$$

$$ARL = \frac{-\ln \alpha_0}{\mu'_1 \ln\left(\frac{\mu'_1}{\mu'_0}\right) - \left(\mu'_1 - \mu'_0\right)} \tag{3.4}$$

4. DISCUSSION OF RESULTS AND CONCLUSIONS

Since we have inspection with error, $\mu' = r\,\mu+\mu_r$.. We have considered four cases $(r, \mu_r) = (1,0),(1,2),(0.8,0)$,and $(0.8,2)$. The first case corresponds to CSCC without inspection error while the other three correspond to different inspection errors.In the CSCC on Poisson variable we must decide on μ_0, μ_1 and α_0. Table- I and II present the value of d and θ needed to construct the control limits. Each table has been entered with the appropriate values of α_0, μ_0 and μ_1. Table-III gives the ARL for CSCC. For ARL, the calculation is based on the assumption that μ_1 is true mean.

From Table-I one can see that error rate of (1,2) results approximately same values of d as of (1,0). This occurs because r = 1 means that none of the non conformities were missed while $\mu_f = 2$ means that there is on the average, two false alarms per part, which is obvious in any .sampling procedure.

An error rate of (0.8,0) results in a big difference in the values of d as compared to (1,0). For r = 0.8 means the inspector is finding only 80% of the non conformities and for different errors rates, the value of d decreases as the value of μ_0 and μ_1 increases.

An error rate of (0.8,2) gives the almost same value of d as for (0.8,0) but the shifts in d are seen when error rates are (1,0) and (1,2) as compared to error rate (0.8,2) and (0.8,0).

Table-I shows that for fixed α_0 the values of d increases as error rates increases. Table-II shows that the angle of the mask (θ) of CSCC increases when the rate (1,2) changes to (0.8,2) and decreases if the rate is (0.8,0) as compared to true observations i.e. error rate (1,0). It is seen from the Table-III that the values of ARL increases as the error rates are (0.8,0), (1,2) and (0.8,2) respectively.

In this paper we have seen that both types of error, i.e. failing to note a nonconformity and noting one where none exists, seriously affect the control limits d, and ARL from that which would be obtained under error free inspection. This statement is especially true when one is setting up CSCC on Poisson variable based on a target value of μ. In any situation, every effort should be made to minimize both types of inspection errors. Here we have assumed a constant r (and μ_f) for the entire inspection scheme but in a reality, r (and μ_f) could vary by inspector, type of nonconformity, shift, etc. Further one should not increase false alarms to compensate for missed non conformities since the cost for such a scheme is high. Both types of errors are to be avoided, but one should not attempt to eliminate one at the cost of the other.

TABLE - I

Control Limits of CSCC for Poisson Variable under Inspection Error (Values of d)

μ_0	μ_1	$(r, \mu_r) = (1,0)$			$(r, \mu_r) = (1,2)$			$(r, \mu_r) = (0.8,0)$			$(r, \mu_r) = (0.8,2)$		
		α_0			α_0			α_0			α_0		
		0.05	0.01	0.001	0.05	0.01	0.001	0.05	0.01	0.001	0.05	0.01	0.001
0.5	1	6.00	9.21	13.82	6.00	9.21	13.82	7.49	11.51	17.27	7.49	11.51	17.27
1.0	2	3.00	4.61	6.91	3.00	4.61	6.91	3.74	5.76	8.63	3.74	5.76	8.63
2	4	1.50	2.30	3.45	1.50	2.30	3.45	1.87	2.88	4.32	1.87	2.88	4.32
3	6	1.00	1.54	2.30	1.00	1.54	2.30	1.25	1.92	2.88	1.25	1.92	2.88
5	10	0.60	1.92	1.38	0.60	1.92	1.38	0.75	1.15	1.73	0.75	1.15	1.73

TABLE - II

Control Limits of CSCC for Poisson Variable under Inspection Error (Values of θ)

μ_0	μ_1	$(r, \mu_r) = (1,0)$	$(r, \mu_r) = (1,2)$	$(r, \mu_r) = (0.8,0)$	$(r, \mu_r) = (0.8,2)$
0.5	1	$35^0\ 48'$	$69^0\ 57'$	$29^0\ 59'$	$68^0\ 55'$
1.0	2	$55^0\ 16'$	$73^0\ 56'$	$49^0\ 5'$	$72^0\ 33'$
2	4	$70^0\ 35'$	$78^0\ 32'$	$66^0\ 34'$	$77^0\ 3'$
3	6	$76^0\ 59'$	$81^0\ 5'$	$73^0\ 53'$	$79^0\ 43'$
5	10	$82^0\ 6'$	$83^0\ 50'$	$80^0\ 10'$	$82^0\ 43'$

TABLE - III

Values of ARL's

μ_0	μ_1	$(r, \mu_r) = (1,0)$		$(r, \mu_r) = (1,2)$		$(r, \mu_r) = (0.8,0)$		$(r, \mu_r) = (0.8,2)$	
		α_0		α_0		α_0		α_0	
		0.05	0.001	0.05	0.001	0.05	0.001	0.05	0.001
0	1	15.5	35.8	63.79	147.08	19.38	44.71	75.63	174.39
1	2	7.8	17.9	60.57	139.67	9.69	22.35	33.78	77.88
2	4	3.9	8.9	6.92	15.96	4.85	11.18	10.12	23.32
5	10	1.6	3.6	2.08	4.80	1.94	4.47	2.66	6.12

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COMMON FIXED POINTS OF COMMUTING MAPPINGS IN D-METRIC SPACES

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ABSTRACT. The aim of this paper is to prove some common fixed point theorems for commuting mappings in generalized metric spaces.

Key words. D-metric space, commuting maps, common fixed point.

AMS Subject Classification. Primary 54H25, Secondary 47H10.

1. INTRODUCTION

Motivated by the measure of nearness between two or more objects with respect to a specific characteristic, called the parameter of the nearness, Dhage [1, 2] introduced a new structure of a generalized metric space (or D-metric space) and proved some fixed point theorems for maps satisfying certain contractive conditions. Generalized metric D is a slight variant of 2-metric.

A generalized metric D is a function $D : X\times X\times X\to R$ where X is a nonempty set satisfying.
(i) $D(x, y, z)\geq 0$ for all x, y, z in X and equality holds iff $x = y = z$,
(ii) $D(x, y, z) = D(y, x, z) = \ldots$ (Symmetry),
(iii) $D(x, y, z)\leq D(x, y, a) + D(x, a, z) + D(a, y, z)$ (Rectangle inequality),
for all x, y, z, a in X.

Pair (X,D) is called generalized metric space (or D-metric space). Geometrically a D-metric $D(x, y, z)$ represents the perimeter of a triangle whose vertices are x, y, z. The metric D is continuous function. Some specific examples of D-metric spaces appear in [1].

In this paper we establish some common fixed point theorems for four commuting mappings satisfying the contractive conditions analogous to [2].

Throughout the discussion (X, D) stands for Generalized metric space. For the symbols, terminology and basic properties of D-metric spaces the reader is referred to [1, 2]. For the sake of convenience, we give some lemmas cited in [2].

Lemma 1.1:[2] (D-Cauchy principle). If X is (S, T)-orbitally bounded D-metric space and $\{x_n\} \subseteq O(S, T; x)$, $x \in X$ satisfying :

$$D(x_n, x_{n+1}, x_m) \leq \phi^n(k), \text{ for all } m > n \in N,$$

where k is D-bound and $\phi : R^+ \to R^+$ satisfies $\Sigma_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t \in R^+$. Then $\{x_n\}$ is D-Cauchy.

Let Φ be a family of all continuous, non-decreasing functions $\phi: R^+ \to R^+$ such that $\phi(t) < t$ for $t > 0$ and $\Sigma_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t \in R^+$

Lemma 1.2 : [2] Let $\phi \in \Phi$ then $\lim_{n \to \infty} \phi^n(t) = 0$ for each $t > 0$ and $\phi^n(0) = 0$ for all $n \in N$.

2. RESULTS.

Theorem 2.1: Let A, B, S and T be self maps of (S, T)-orbitally complete and (S, T)-orbitally bounded D-metric space X satisfying:

(2.1.1) $A(X) \subseteq T(X)$; $B(X) \subseteq S(X)$,

(2.1.2) one of A, B, S or T is continuous,

(2.1.3) (A, S) and (B, T) are commuting pairs of maps,

(2.1.4) for all $x, y \in X$, $z \in \overline{O(S, T; x) \cup O(T, S; x)}$,

$$D(Ax, By, z) \leq \phi(\max\{D(Sx, Ty, z),\ D(Ax, Ty, z),\ D(By, Sx, z)\}),$$

where $0 \leq \beta < 1/3$, $\phi \in \Phi$. Then A, B, S and T have a unique common fixed point in X.

Proof . Let x_0 be an arbitrary point in X. By virtue of (2.1.1), for x_1, x_2 in X, $Ax_0 = Tx_1$, $Bx_1 = Sx_2$. Inductively construct a sequence $\{y_n\}$ in X such that

$$y_{2n+1} = Ax_{2n} = Tx_{2n+1} \; ; \; y_{2n} = Bx_{2n-1} = Sx_{2n} \; ; \; n = 0, 1, 2, \ldots .$$

Using (2.1.4) for any $m \geq 2$, we have

$$\begin{aligned}
D(y_1, y_2, y_m) &= D(Ax_0, Bx_1, y_m), \\
&\leq (\max\{D(Sx_0, Tx_1, y_m), D(Ax_0, Tx_1, y_m), D(Bx_1, Sx_0, y_m)\}), \\
&= (\max\{D(y_0, y_1, y_m), D(y_1, y_1, y_m), D(y_2, y_0, y_m)\}), \\
&= [\max_{\substack{0 \leq a \leq 1 \\ 1 \leq b \leq 2}} D(y_a, y_b, y_m)], \\
&\leq \phi(k), \text{ where } k \text{ is a D-bound of } O(S, T; x).
\end{aligned}$$

Similarly, for any $m \geq 3$, we have

$$\begin{aligned}
D(y_2, y_3, y_m) &= D(Ax_2, Bx_1, y_m), \\
&\leq \phi(\max\{D(Sx_2, Tx_1, y_m), D(Ax_2, Tx_1, y_m), D(Bx_1, Sx_2, y_m)\}), \\
&= (\max\{D(y_1, y_2, y_m), D(y_1, y_3, y_m), D(y_2, y_2, y_m)\}).
\end{aligned}$$

Now there are three cases :

Case I. $D(y_2, y_3, y_m) \leq \phi\{D(y_1, y_2, y_m)\} \leq \phi(\phi(k)) = \phi^2(k)$.

Case II. $D(y_2, y_3, y_m) \leq \phi\{\beta\, D(y_2, y_2, y_m)\}$

$$\begin{aligned}
&\leq \phi\, [\beta\{D(y_1, y_2, y_2) + D(y_1, y_2, y_m) + D(y_1, y_2, y_m)\}], \\
&\leq \phi(3\beta\phi(k)) \leq \phi^2(k), \text{ (since } 3\beta \leq 1).
\end{aligned}$$

Case III. $D(y_2, y_3, y_m) \leq \phi(\beta(D(y_1, y_3, y_m))$ ... ... (

But $D(y_1, y_3, y_m) \leq \beta\, [D(y_1, y_3, y_2) + D(y_1, y_2, y_m) + D(y_2, y_3, y_m)]$,

$$\leq 2\beta\phi(k) + \beta\, D(y_2, y_3, y_m). \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

Hence from (i) and (ii), we get

$$\begin{aligned}
D(y_2, y_3, y_m) &\leq \phi\{2\beta\phi(k) + \beta D(y_2, y_3, y_m)\}, \\
&< 2\beta\, \phi(k) + \beta D(y_2, y_3, y_m),
\end{aligned}$$

or,

$$D(y_2, y_3, y_m) < 2\beta\, \phi(k)/(1-\beta) \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

From (ii) and (iii), we have

$$\begin{aligned}
D(y_1, y_3, y_m) &< 2\beta\, \phi(k) + 2\beta^2\, \phi(k)/(1-\beta\,) = 2\beta\, \phi(k)/(1-\beta\,), \\
&\leq \phi(k), \text{ (since } 3\beta \leq 1). \text{ Therefore}
\end{aligned}$$

$$\phi(\beta D(y_1, y_3, y_m) \leq \phi(\phi(k)) = \phi^2(k) \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

Consequently from (i) and (iv), $D(y_2, y_3, y_m) \leq \phi^2(k)$. Hence in all cas we have $D(y_2, y_3, y_m) \leq \phi^2(k)$. Inductively for $m \geq n+1$, we have

$$D(y_n, y_{n+1}, y_m) \leq \phi^n(k).$$

Hence from Lemma 1.1, $\{y_n\}$ is D-Cauchy. Now by (S, T)-orbitally completeness of X, $\{y_n\}$ and also its subsequences $\{Ax_{2n}\}$, $\{Bx_{2n-1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ converge to some u in X.

Suppose that the map S is continuous then we have $\lim_{n\to\infty} SAx_{2n} = \lim_{n\to\infty} SSx_{2n} = Su$. Now by commutativity of (A, S), we have $\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} SAx_n = Su$. Using (2.1.4), we have

$D(ASx_{2n}, Bx_{2n+1}, u) \leq \phi(\max\{D(SSx_{2n}, Tx_{2n+1}, u), \beta D(ASx_{2n}, Tx_{2n+1}, u), \beta D(Bx_{2n+1}, Sx_{2n}, u)\}.$

Letting $n \to \infty$, we get

$$D(Su, u, u) \leq \phi\,(\max\{D(Su, u, u), \beta D(Su, u, u), \beta D(u, Su, u)\}),$$
$$= (D(Su, u, u)),$$ which yields $Su = u$.

Further,

$D(Au, Bx_{2n-1}, u) \quad \leq \phi\,(\max\{D(Su, Tx_{2n-1}, u), \beta D(Au, Tx_{2n-1}, u), \beta D(Bx_{2n-1}, Su, u)\}).$

Letting $n \to \infty$, we get

$$D(Au, u, u) \leq \phi\,(\max\{D(u, u, u), \beta D(Au, u, u), \beta D(u, u, u)\},$$
$$\leq \phi(D(Au, u, u)),$$ which yields $Au = u$.

Since $Au = u$ and $A(X) \subseteq T(X)$ there exists $v \in X$ such that $u = Tv. = Su$. Using (2.1.4), we have

$$D(u, Bv, u) = D(Au, Bv, u),$$
$$\leq \phi\,(\max\{D(Su, Tv, u), \beta D(Au, Tv, u), \beta D(Bv, Su, u)\},$$
$$= \phi\,(\max\{D(u, u, u), \beta D(u, u, u), \beta D(Bv, u, u)\},$$
$$\leq \phi(D(Bv, u, u)),$$ which yields $Bv = u$. Thus $u = Bv = Tv$.

From (2.1.3), $BTv = TBv$ or $Bu = Tu$. Using (2.1.4), we have

$$D(u, Tu, u) = D(Au, Bu, u),$$
$$\leq \phi(\max\{D(Su, Tu, u), \beta D(Au, Tu, u), \beta D(Bu, Su, u)\},$$
$$= \phi(\max\{D(u, Tu, u), \beta D(u, u, u), \beta D(Tu, u, u)\},$$
$$\leq \phi\,(D(u, Tu, u)),$$ which yields $u = Tu = Bu$.

Hence $Au = Bu = Su = Tu = u$, i.e. u is the common fixed point of A, B, S and T.

Now for the uniqueness of u, let w be another common fixed point of A, B, S and T. Then from (2.1.4), we have

$$D(u, w, u) = D(Au, Bw, u),$$
$$\leq \phi\,(\max\{D(Su, Tw, u), \beta D(Au, Tw, u), \beta D(Bw, Su, u)\},$$
$$= \phi\,(\max\{D(u, w, u), \beta D(u, w, u), \beta D(w, u, u)\},$$
$$\leq \phi\,(D(u, w, u)),$$ which yields $u = w$. This completes the proof.

As a slight generalization of Theorem 2.1, we have the following result.

Theorem *2.2* : Let A, B, S and T be self maps of (S^s, T^t)-orbitally complete and (S^s, T^t)-orbitally bounded D-metric space X satisfying (2.1.2), (2.1.3) and

(2.2.1) $A^a(X) \subseteq T^t(X)$; $B^b(X) \subseteq S^s(X)$,

(2.2.2) for all $x, y \in X$, $z \in \overline{O(S^s, T^t, x)} \cup \overline{O(T^t, S^s, y)}$,

$$D(A^a x, B^b y, z) \le \phi\,(\max\{D(S^s x, T^t y, z), \beta D(A^a x, T^t y, z), \beta D(B^b y, S^s x, z)\},$$

where a, b, s and t are positive integer, $0 \le \beta < 1/3$ and $\phi \in \Phi$. Then A, B, S and T have a unique common fixed point in X.

Proof. Since (A, S) and (B, T) are commuting pairs of maps so (A^a, S^s) and (B^b, T^t) are also commuting pairs of maps. Therefore by Theorem (2.1) there exists unique u inX such that $u = A^a u = B^b u = S^s u = T^t u$.

Now $Au = A(A^a u) = A^a(Au)$; $Au = A(S^s u) = S^s(Au)$, i.e. Au is the common fixed point of A^a and S^s. Similarly Bu is the common fixed point of B^b and T^t.

Using (2.2.2), we have (put x = Au , y = Bu)

$$\begin{aligned} D(Au, Bu, u) &= D(A^a x, B^b y, u), \\ &\le \phi(\max\{D(S^s x, T^t y, u), \beta D(A^a x, T^t y, u), \beta D(B^b y, S^s x, u)\}, \\ &= \phi(\max\{D(Au, Bu, u), \beta D(Au, Bu, u), \beta D(Bu, Au, u)\}, \\ &\le \phi(D(Au, Bu, u)) \text{ which yields } Au = Bu = u. \end{aligned}$$

Further $Su = S(A^a u) = A^a(Su)$; $Su = S(S^s u) = S^s(Su)$ i.e. Su is the common fixed point of A^a and S^s. Similarly Tu is the common fixed point of B^b and T^t.

Using (2.2.2), we have (put x = Su, y = Tu)

$$\begin{aligned} D(Su, Tu, u) &= D(A^a x, B^b y, u), \\ &\le \phi(\max\{D(S^s x, T^t y, u), \beta D(A^a x, T^t y, u), \beta D(B^b y, S^s x, u)\}, \\ &= \phi(\max\{D(Su, Tu, u), \beta D(Su, Tu, u), \beta D(Tu, Su, u)\}, \\ &\le \phi\,(D(Su, Tu, u), \text{ which yields } Su = Tu = u. \end{aligned}$$

Hence Au = Bu = Su = Tu = u. This completes the proof.

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SECURITY ISSUES AND PRACTICES ON THE INTRANET

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ABSTRACT. By definition an intranet is internal to an organisation it means ; outsiders can't access it. Intranet security is, then, a multifaceted issue, with both opportunities and dangers, especially if network is part of the Internet. Proper and effective security measures can be used to widen the scope of an intranet. In this paper we are presenting some strategies and practices to secure the intranet from unauthorised access.

KEYWORDS. Intranet, Internet, Packets, Gateways, IP Address, Firewalls.

1. INTRODUCTION

"The security of the networking environment is critical for today's corporations. Issues such as reliably identifying remote-access users, checking authorisations, data-transfer security and database security are of paramount importance" [2]. Generally most of the people view computer and network security only in terms of restricting access to services. One major view of network security is " that which is not expressly permitted is denied". Intranet security can be an enabler, enriching an intranet with services and resources which generally we would not be able to provide. Thus rather thinking only in terms of restricting access to services a different and more positive security policy might be described as "that which is not expressly denied is permitted." This does not mean that we should not take care of cautions and make everything available to users on an intranet. There are many things to consider when placing sensitive business data out on an intranet. This paper takes the latter approach, presenting intranet security in terms of its opportunities for adding value to an intranet. Providing proper security, increases an organisation's ability to use the important collaborative aspects of an intranet. Though level of security may change from organisation to organisation. Businesses in which confidentiality and discretion are the norm in handling proprietary information and corporate intellectual property have different needs than a college or university.

2. SECURITY THROUGH ACCESS POLICIES

One of the most important parameter of network security is to have a written policy. It is generally seen that most of the intranet administrator overlook this and due to this ignorance most of the security problems arises. A policy is a document that describes what we are trying to protect, as well as describing what should and shouldn't be done over the network. Having a policy allows us to measure how effectively we have implemented procedures. Policies also help us to explain to users and management why some things need to be done differently or not at all.

2.1. TIER ARCHITECTURE FOR WRITING POLICIES

Writing policies is a complicated task because it requires to keep the ideas general enough to be flexible but specific enough to be useful. There are normally two different parts to the policy. The first part is the security portion, which describes what needs to be protected and how to protect it. The second part is the access part. This part describes what can be done over the network. The first step in writing the security portion is to define what needs to be protected. This is the hardest part to determine because most users don't know what they use or how important it is. It is often better to work with each group, determine what they use for files and programs, and compile a list of "who uses what". Next we need to determine who shouldn't be able to get to what. Often security is implemented in layers or tiers. The top tier is for very sensitive machines such as those that create accounts or that enforce security. The second tier of information should be allowed to be changed by only certain groups. It may need to be viewed by more than one group. Define what those groups are and what access they need. The third tier is usually readable by everyone and writable by only certain groups. The last tier is information that can be written or read by everyone. The Internet usually falls under one of these tiers. Some companies may only allow certain people or groups of people to post information. Such type of good practices helps to eliminate inaccurate information from being sent out from the company.

2.2. USER'S AWARENESS

One of the most important steps in allowing employees to access the Internet or Intranet is to teach them how to use it. There should be a document or required training class that employees must take before using the Intranet. This class or document needs to cover a few issues like how the software works, what is hypertext, how it works, what is available and how to find that which is not readily available. And the knowledge of any Internet tools the company uses like: News readers, WWW browsers, FTP clients, E-mail, etc. If the document, or class, covers these basics and teaches the users how to act responsibly on the Intranet, many problems can be avoided.

3. SECURITY THROUGH OBSCURITY : MAKE IT HARD TO FIND

One of the easiest ways to prevent casual users from accessing the Intranet is to make it hard to find. This will not prevent someone who is trying to break into a site, but it may keep out some people. Obscurity is a good way to make intruders spend time while they are trying to get in the system and so that it may alert someone who is responsible for the security of the system. The best defense, though, is to use a combination of security measures, such as server security or firewalls, in conjunction with obscurity. This could be very useful way of altering the person concerned before the intruders can enter into the system. There can be a few ways to hide the intranet. The following practices can be very useful in securing an intranet.

3.1. USING NON-STANDARD PORTS

We know that most of the common applications make use of standard port number and those applications are generally known by those port numbers. For example for some specific application the standard port is 80. Using a different port will make it harder to find. From network security point of view it is unfortunate that there are many programs available that can serach for open ports on a machine or a range of machines. It is assumed that port scanning software is a hacker's friend, but it can also be a network manager's freind, if network manager can use it smartly and properly. Port scanning software can detect unauthorized servers running that can cause securing problems.

3.2. USING HARD TO GUESS NAMES

Generally, while teaching programming conventins we teach to the students to give appropriate name to the programs and applications but these conventions can create problems as far as network security is concerned. For example it is very common for most of the organizations use WWW for the Web server machine name. Using something different can make it harder to find

3.3.HIDING SERVER'S NAME

This can be done by not listing it in the DNS tables for a particular site, and not using it to browse the Web, send e-mail, or post to Usenet.

4. SECURING SERVER

Simply hiding the site is not enough. Most WWW servers offer a layer of protection called access controls. These access controls may be used to allow us to define IP address ranges that can retrieve documents from your Web server. Most Web servers also allow us to specify a username and password before allowing any documents to be retrieved. There are two security models which we can use to secure a Web server:" All that is not allowed is denied. This means that we start out denying everyone access and only allowing certain machines access. This can usually be done by placing an * (or all) in the deny field and placing the local network number in the allow field." All that is not denied is allowed. This means that we allow everyone and deny those sites that are known to be bad. This would be done by placing an * (or all) in the allow field and listing out any site that shouldn't be allowed access to the deny field.

Security experts agree that it is best to deny everything and only allow in what is needed. This reduces the chance of the administrator overlooking something. In an Intranet it is better to deny everyone and allow local IP addresses, rather then try to list everyone who can't see what is on the Intranet.

4.1. SERVER SECURITY BY RESTRICTING IP ADDRESS

Almost all Web servers have an access list (Which is known as Access Control List- ACL) that defines what machines or networks are allowed to retrieve documents or submit forms. This access list is usually made up of a list of allow and deny fields. Restricting access based on the IP address of the requesting machine is a good start toward security; however, that security is only as good as the security of the IP protocol.

4.2. USERNAMES AND PASSWORDS

Using IP addresses as security protection is a good start but doesn't handle all cases. The most obvious case is when multiple people use the same machine but only one person should have access to an Intranet. This can happen if users use Internet Service Providers (ISP's) to gain access when they are out of the office. ISP's generally allow multiple people to log in to one machine or allow dial-up users to share an IP address. Either way it can be disastrous to a security policy based on IP addresses for access decisions. In this case, it is required to use usernames and passwords. Most Web servers allow username security. When a user encounters a page that is protected, a box appears asking for username. After the username is entered, the password is required. Once the password has been entered, it is checked to make sure it is the correct one. If so, the document is sent to the user; otherwise, an error is sent back. Users usually only need to authenticate once per site, per session.

5. SECURITY THROUGH FIREWALLS

The Web server software security is just not enough, because there are too many other ways to get information, such as FTP, TFTP, or shared file systems. Firewalls are a system or group of systems that enforce a policy between two networks. A firewall can be anything from a set of filtering rules set up on the router between you and the Internet to an elaborate application gateway consisting of one or more specially configured computers that control access. "Firewalls permit desired services coming from the outside, such as Internet e-mail, to pass. In addition, most firewalls now allow access to the World Wide Web from inside the protected networks. In most cases one of the networks is the Internet; however, firewalls can be placed between any two networks." [4]. Firewalls are of two types (i) Network-level and (ii) Application-level firewalls. Both have their strengths and weaknesses. It is possible and often desirable to use a combination of the two type of firewalls, depending on security requirements.

5.1 NETWORK LEVEL FIREWALLS

Network-level firewalls are devices or systems that look at the IP packet header and decide whether the address and port are allowed to pass through or not. It does not look at the data portion of the packet and does not understand what is going through. The most common network-level firewall is simply a router with an access list defined on it. Machines with two network cards can also be used using special filtering software.

5.1.1. ROUTER ACCESS LISTS

Many routers allow the administrator to define an access policy. These access policies tell the router what packets are allowed to pass and which ones are to be dropped. A simple access policy can be as follows:

Rule 1: Allow port 25 (e-mail) from anyone into the mailhost.
Rule 2: Allow port 80 (WWW) from anyone into the external Web server.
Rule 3: Allow users from 123.234.123 network log in to port 23 (telnet) to mailhost.
Rule 4: Don't allow anything else in.

This simple access list allows anyone to send e-mail to the mailhost (Rule 1), and allows customers to get to the external WWW server (Rule 2). It also allows people on the network 123.234.123 to log in to the machine mailhost. The most important line is the last one, which says no one else can get in. Access lists can l be much more complex than this. Some routers allow us to define what to do if a packet is refused. Access lists generally can be placed on the external port, on the internal port, or both, depending on the router. Allowing traffic to a specific set of machines is commonly referred to as a screened-host firewall. This means that traffic is allowed to a specific host or set of hosts but only if it fits through the screen.

5.1.2. SECURITY BY FILTERING PACKETS

A router-based packet-filtering firewall is an effective way of protecting an enterprise network from unauthorised access. "[1]. Very few practices to this problem have been proposed in the literature, and none is completely satisfactory. Packet filters are usually machines with two network cards running special software to allow or disallow packets based on their address and destination port. Packet filters are similar to routers and can be used to build screened-host firewalls. In this paper we are discussing some practices in next few sections related to router-based packet-filtering firewall.

5.2. APPLICATION LEVEL FIREWALLS

Application-level firewalls look at the packet and decide what the packet is trying to do and whether it is permitted or not. Application-level firewalls are very secure hosts with two network cards: one set up on the internal side and one set up to run on the hostile side. This bastion host runs proxy servers for each protocol that needs to get through the firewall. This allows no traffic to pass between the two networks and the proxy must instead interpret and relay data back and forth.

5.2.1. ADDRESS TRANSLATION

Because application-level firewalls relay messages instead of passing traffic through, they can be used as Network Address Translators (NAT's). NAT's are useful for IP networks that are using unregistered address ranges or reserved ranges. " With the explosive growth of the Internet it was realized that there were not enough addresses for everyone who used IP to have a registered address. As a result, RFC 1597 was written. It advised setting aside different ranges of addresses not to be used on the Internet, but to be used only on internal networks." [3]. Because the Internet is designed not to route traffic to any reserved IP address ranges, using them for internal networks can help to prevent attacks from the Internet. This added security is not without a cost, though, because every external connection may require a network address translator to work properly.

5.2.2. FIREWALL CONFIGURATIONS :

Firewalls are configured in different ways, depending on how an organisation decides to balance the cost versus security issue. The most secure firewalls often use a combination of security layers to offer the most protection. There are three types of basic firewall configurations in use today. They are:

5.2.2.1. DUAL-HOMED GATEWAYS

Dual-homed gateways are hosts configured to support two network interfaces. One faces in toward the secure network and one faces out toward the hostile network. They are configured not to route traffic between them. The dual-homed machine is also called a bastion host. This machine needs to be as secure as possible, because it is the way a hacker will try to get in from the Internet. The bastion host is the host that would normally run the external WWW server and FTP server as well as act as a mailhost for the Internet. Dual-homed gateways are the cheapest of the three basic·firewalls.

5.2.2.1. SCREENED-HOST GATEWAYS

Screened-host gateways are built using a screening router to block traffic to the internal network and to allow traffic to a bastion host. This host has one network interface as opposed to the dual-homed gateway that has two. Because most Internet service providers provide a router at the site, it is easy and often economical to use this configuration. Screened hosts have the advantage of being able to allow certain applications into the network easily.

5.2.2.3. SCREENED-SUBNET GATEWAYS

A screened sub-net is made up of two screening routers with the bastion in the middle. This makes a small isolated network between the secure and hostile networks. The bastion host sits in this isolated network. "It is possible to have multiple hosts in this isolated network, also called Demilitarised Zone or DMZ. " This configuration is the most expensive but it is also the most secure because two devices need to fail to allow the network to be exposed. Each configuration has a different trade-off among between security, cost, and performance. For example, dual-homed gateways are easier to configure and set up then screened hosts, but at a slight loss in security.

6. GENERAL SECURITY

Many times network administrators put a firewall in place and assume that they are safe. However, this is very rarely the case. This is similar to putting a large steel door on the front of our house, and not paying attention to the glass windows scattered around the building.

6.1. ATTACK THROUGH A REMOTE NETWORK

Many companies have modems on their employees' machines. These can be for transferring files, connecting to online services, or might not be used at all. Many of these modems might be set to Auto Answer when called. Auto Answer modems by themselves aren't a problem, especially on PCs that usually don't allow logins. However, UNIX machines are normally configured to allow remote logins on modem ports. This allows people in without going through the firewall. UNIX machines aren't the only ones with remote capabilities, though. Many PC modem packages allow a remote user to take over a PC just as if he were sitting at the keyboard. This includes any network access that the machine may have. Even worse than a login session is a PPP or SLIP connection. Many newer systems, such as Windows 95 or Windows NT, come with PPP software built in. It is possible for a user, knowingly or not, to enable this access. This allows the hacker to access any machine on the network virtually untraced. It is important for a security administrator to be aware of these potential security holes. Hackers have access to many phone number scanners. These scanners work much like port scanners or network scanners. They start dialing numbers until they get a modem tone and then try to break in. Knowing which phone lines are connected to modems and securing them is as important as configuring a firewall.

In addition to the Internet link an organisation may have other remote networks that can be a source of attack. These may be something like : telenet, or a dedicated or dial-up link to another company. Regardless of where these links go, we should always assume that they connect to a hostile network and security should be in place on them. This may include a firewall or another safety device.

7. CONCLUSION

In any organisation having its own intranet, it is the responsibility of network administrator or network decision makers to determine the level of security. Decision makers will have to make sure that they are putting the valuable information on intranet in a secure fashion. Poor security practices can't protect an Intranet through unauthorised access and attack. In other words, network administrator should always try to use good security practices. Our study shows both technical solutions and policy decisions that can be implemented to make things safer and more productive. Thus in this paper we have tried to present some important and effective security practices to provide secure access to anyone to an intranet from outside or inside the organisation.

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COMMON FIXED POINTS OF FOUR MAPPINGS OF A FUZZY METRIC SPACES

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ABSTRACT. The aim of this paper is to obtain common fixed point theorem for four mappings of fuzzy metric spaces.

Key words. R-weakly commuting maps, compatible maps, fuzzy metric spaces.

A.M.S.(MOS) subject classification 4 7 H 10.

1. INTRODUCTION

The contraction principle in the setting of fuzzy metric spaces is introduced by Kramosil and Michalek [2] and established by Grabiec [1] Subramanyam [4] has further generalized Grabiec result for a pair of commuting mappings. Vasuki [5] defined the R- weak commutativity of mappings of fuzzy metric spaces and proved the fuzzy version of Pant's theorem.

Recently Singh and Chauhan [3] have introduced the concept of compatible mappings of fuzzy metric spaces and then proved a common fixed point theorem for two pairs of compatible mappings out of which two mapping are continuous. Using the concept of compatibility and R- weak commutativity of mappings together with reciprocal continuity of maps of fuzzy metric spaces, we have proved a common fixed point theorem. For this purpose we need the following definitions.

Definition 1. The 3-tuple (X, M, $*$) is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$, satisfying following conditions:

(i) $M(x,y,t) > 0$,

(ii) $M(x,y,t) = 1$ if and only if $x = y$,

(iii) $M(x,y,t) = M(y,x,t)$,

(iv) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$,

(v) $M(x,y,.): (0,\infty) \to [0,1]$ is continuous, for all $x, y, z \in X$ and $t, s > 0$.

Definition 2. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is a Cauchy sequence if and only if for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1-\varepsilon$ for all $n, m \geq n_0$.

Definition 3. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ converges to x if and only if for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x, t) > 1-\varepsilon$ for all $n \geq n_0$.

Definition 4. Fuzzy metric space $(X,M,*)$ is said to be complete if every Cauchy sequence in $(X,M,*)$ is a convergent sequence.

Definition 5. Two mappings f and g of a fuzzy metric space $(X, M, *)$ into itself are said to be weakly commuting if $M(fgx, gfx, t) \geq M(fx, gx, t)$ for each x in X.

Definition 6. The mappings f and g of a fuzzy metric space $(X, M, *)$ into itself are said to be R-weakly commuting, provided there exists some positive real number R such that $M(fgx, gfx, t) \geq M(fx, gx, t/R)$ for each x in X.

Definition 7. Self mappings F and G of a fuzzy metric space $(X, M, *)$ are said to be compatible iff $M(FGx_n, GFx_n, t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Fx_n, Gx_n \to y$ for some y in X.

Definition 8. Let A and S be self mappings of a fuzzy metric space $(X,M,*)$. We will call A and S to be reciprocally continuous if $\lim_{n \to \infty} ASx_n = Ap$ and $\lim_{n \to \infty} SAx_n = Sp$ whenever $\{x_n\}$ is a sequence such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = p$ for some p in X.

If A and S are continuous then they are obviously reciprocally continuous. But the converse need not be true.

Singh and Chauhan [3] have proved the following theorem .

Theorem 1. Let A,B,S and T be self maps of a complete fuzzy metric space $(X,M,*)$ with continuous t-norm $*$ defined by $a * b = \min\{a,b\}$, $a,b \in [0,1]$, satisfying the following conditions

(i) $A(X) \subset T(X)$, $B(X) \subset S(X)$,

(ii) S and T are continuous,

(iii) [A,S], [B,T] are compatible pairs of maps,

(iv) for all x,y in X, $k \in (0,1)$, $t > 0$
$M(Ax, By, kt) \geq \min\{M(Sx,Ty,t), M(Ax,Sx,t), M(By,Ty,t), M(By,Sx,2t), M(Ax,Ty,t)\}$,

(v) for all x,y in X, $\lim M(x,y,t) \to 1$, as $t \to \infty$.

Then A,B,S and T have a unique common fixed point in X.

Motivated by Singh. and Chauhan [3], we shall prove the following.

Theorem 2. Let A,B,S and T be self maps of a complete fuzzy metric space $(X,M,*)$
with continuous t-norm $*$ defined by $a*b = \min\{a,b\}$, $a,b \in [0,1]$, satisfying the following conditions

(i) $A(X) \subset T(X)$, $B(X) \subset S(X)$,

(ii) [A,S], [B,T] are pointwise R-weakly commuting pairs of maps,

(iii) [A,S] or [B,T] is compatible pair of reciprocally continuous maps,

(iv) for all x,y in X, $k \in (0,1)$, $t > 0$

$$M^3(Ax, By, kt) \geq \max\{M^3(Sx, Ty, t), M^3(Ax, Sx, t), M^3(By, Ty, t), M(Ax, Ty, 2t)M(By, Sx, 2t)M^2(By, Ty, t)\}, \tag{1}$$

(v) for all x,y in X, $\lim M(x,y,t) \to 1$, as $t \to \infty$.

Then A, B, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Construct a sequence $\{y_n\}$ in X such that

$$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2} \text{ and } y_{2n} = Sx_{2n} = Bx_{2n-1};\ n=1,2,3,\ldots$$

Now using (iv), we have

$$M^3(y_{2n+1}, y_{2n+2}, kt) = M^3(Ax_{2n}, Bx_{2n+1}, kt)$$

$$\begin{aligned}
&\geq \max\{M^3(Sx_{2n}, Tx_{2n+1}, t), M^3(Ax_{2n}, Sx_{2n}, t), M^3(Bx_{2n+1}, Tx_{2n+1}, t),\\
&\qquad M(Ax_{2n}, Tx_{2n+1}, 2t) M(Bx_{2n+1}, Sx_{2n}, 2t) M^2(Bx_{2n+1}, Tx_{2n+1}, t)\}\\
&= \max\{M^3(y_{2n}, y_{2n+1}, t), M^3(y_{2n}, y_{2n+1}, t), M^3(y_{2n+2}, y_{2n+1}, t),\\
&\qquad M(y_{2n+1}, y_{2n+1}, 2t) M(y_{2n+2}, y_{2n}, 2t) M^2(y_{2n+2}, y_{2n+1}, t)\}\\
&\geq \max\{M^3(y_{2n}, y_{2n+1}, t), M^3(y_{2n+2}, y_{2n+1}, t),\\
&\qquad M(y_{2n+2}, y_{2n}, 2t) M^2(y_{2n+2}, y_{2n+1}, t)\}.\\
&\geq \max[M^3(y_{2n}, y_{2n+1}, t), M^3(y_{2n+2}, y_{2n+1}, t), \min\{M(y_{2n+2}, y_{2n+1}, t),\\
&\qquad M(y_{2n+1}, y_{2n}, t)\} M^2(y_{2n+2}, y_{2n+1}, t)].
\end{aligned}$$

$$\Rightarrow M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) \tag{1}$$

Further using (iv), we have

$$\begin{aligned}
M^3(y_{2n}, y_{2n+1}, kt) &= M^3(Bx_{2n-1}, Ax_{2n}, kt)\\
&= M^3(Ax_{2n}, Bx_{2n-1}, kt)\\
&\geq \max\{M^3(Sx_{2n}, Tx_{2n-1}, t), M^3(Ax_{2n}, Sx_{2n}, t), M^3(Bx_{2n-1}, Tx_{2n-1}, t),\\
&\qquad M(Ax_{2n}, Tx_{2n-1}, 2t) M(Bx_{2n-1}, Sx_{2n}, 2t) M^2(Bx_{2n-1}, Tx_{2n-1}, t)\}\\
&= \max\{M^3(y_{2n}, y_{2n-1}, t), M^3(y_{2n}, y_{2n+1}, t), M^3(y_{2n}, y_{2n-1}, t),\\
&\qquad M(y_{2n+1}, y_{2n-1}, 2t) M(y_{2n}, y_{2n}, 2t) M^2(y_{2n}, y_{2n-1}, t)\}\\
&\geq \max[M^3(y_{2n}, y_{2n-1}, t), M^3(y_{2n+1}, y_{2n}, t),\\
&\qquad \min\{M(y_{2n+1}, y_{2n}, t), M(y_{2n}, y_{2n-1}, t)\} M^2(y_{2n}, y_{2n-1}, t)].
\end{aligned}$$

$$\Rightarrow M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}, y_{2n}, t) \tag{2}$$

Using (1) and (2), we have

$$\begin{aligned} M(y_n, y_{n-1}, (1\text{-}k)t / k) &\geq M(y_{n-1}, y_{n-2}, (1\text{-}k)t / k^2) \\ &\geq M(y_{n-2}, y_{n-3}, (1\text{-}k)t / k^3) \\ &\quad \cdots \qquad \cdots \\ &\geq M(y_0, y_1, (1\text{-}k)t / k^n) \qquad \to 1 \text{ as } n \to \infty. \end{aligned}$$

Hence for $t > 0$; $k, \lambda \in (0\ 1)$, we can choose $n_0 \in N$ such that

$$M(y_n, y_{n-1}, (1\text{-}k)t / k) > 1 - \lambda. \quad n \geq n_0. \tag{3}$$

To prove that $\{y_n\}$ is a Cauchy sequence,we claim (4) is true for all $n \geq n_0$. and for every $m \in N$

$$M(y_n, y_{n+m}, t) > 1 - \lambda. \tag{4}$$

From (1),(2) and (3),we have

$$M(y_n, y_{n+1}, t) \geq M(y_n, y_{n-1}, t / k) \geq M(y_n, y_{n-1}, (1\text{-}k)t / k) > 1 - \lambda.$$

Thus result (4) is true for $m = 1$.Further,suppose (4) is true for m. Then we shall show that it is also true for m+1.

Using (1), (2) and definition of t-norm , we have

$$\begin{aligned} M(y_n, y_{n+m+1}, t) &\geq M(y_n, y_{n+m}, t / k) \\ &\geq \min \{M(y_n, y_{n-1}, (1\text{-}k)t / k) , M(y_n, y_{n+m}, t)\} > 1 - \lambda. \end{aligned}$$

Thus (4) is true for m+1 and so it is true for every $m \in N$.

Therefore $\{y_n\}$ is a Cauchy sequence.Since $(X, M, _*)$ is complete, so $\{y_n\}$ converges to some point z in X. Thus $\{Ax_{2n}\}, \{Sx_{2n}\}, \{Bx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ also converges to z .Suppose [A,S] is a compatible pair of reciprocally continuous maps.Then by the definition of reciprocally continuous maps, $ASx_{2n} \to Az$ and $SAx_{2n} \to Sz$ and then the compatibility of A and S yields $\lim_{n \to \infty} M(ASx_{2n}, SAx_{2n}, t) = 1$ i.e. $M(Az, Sz, t) = 1$.Hence $Az = Sz$.

Since $A(X) \subset T(X)$, there exists a point w in X such that $Az = Tw$. Using (iv), we have

$$M^3(Az, Bw, kt) \geq \max\{M^3(Sz, Tw, t), M^3(Az, Sz, t), M^3(Bw, Tw, t),$$
$$M(Az, Tw, 2t)M(Bw, Sz, 2t)M^2(Bw, Tw, t)\}$$
$$= \max\{M^3(Az, Az, t), 1, M^3(Bw, Az, t),$$
$$M(Az, Az, 2t)M(Bw, Az, 2t)M^2(Bw, Az, t)\}$$

or $M^3(Az, Bw, kt) \geq 1$

which implies that $Az = Bw$. Thus $Sz = Az = Tw = Bw$.

Pointwise R-weakly commutativity of A and S implies that there exists $R > 0$ such that $M(ASz, SAz, t) \geq M(Az, Sz, t / R) = 1$

i.e. $ASz = SAz$

and $AAz = ASz = SAz = SSz$.

Similarly pointwise R-weakly commutativity of B and T implies that

$$BBz = BTw = TBw = TTw.$$

Now by (iv), we have

$$M^3(AAz, Az, kt) = M^3(AAz, Bw, kt)$$
$$\geq \max\{M^3(SAz, Tw, t), M^3(AAz, SAz, t), M^3(Bw, Tw, t),$$
$$M(AAz, Tw, 2t)M(Bw, SAz, 2t)M^2(Bw, Tw, t)\}$$
$$= \max\{M^3(AAz, Az, t), 1, M(AAz, Az, 2t)M(Az, AAz, 2t)\}$$

or $M^3(AAz, Az, kt) \geq 1$

$\Rightarrow$ $AAz = Az$. Thus $Az = AAz = SAz$.

Thus Az is a common fixed point of A and S.

Again by (iv), we have

$$M^3(Az, BBw, kt) \geq \max\{M^3(Az, TBw, t), M^3(Az, Sz, t), M^3(BBw, TBw, t),$$
$$M(Az, TBw, 2t)M(BBw, Sz, 2t)M^2(BBw, TBw, t)\}$$
$$= \max\{M^3(Az, BBw, t), 1, M(Az, BBw, 2t)M(BBw, Az, 2t)\}$$

or $M^3(Az, BBw, kt) \geq 1 \Rightarrow Az = BBw$. Thus $Az = BBw = Bw$.

Thus Bw (= Az) is a common fixed point of B and T and hence Az is a common fixed point of A, B, S and T.

To prove uniqueness, let Az_1 be another common fixed point of A, B, S and T. Then, we have

$$\begin{aligned} M^3(Az, Az_1, kt) &= M^3(AAz, BAz_1, kt) \\ &\geq \max\{M^3(SAz, TAz_1, t), M^3(AAz, SAz, t), M^3(BAz_1, TAz_1, t), \\ &\qquad M(AAz, TAz_1, 2t)M(BAz_1, SAz, 2t)\, M^2(BAz_1, TAz_1, t)\} \\ &= \max\{M^3(Az, Az_1, t), M^3(Az, Az, t), M^3(Az_1, Az_1, t), \\ &\qquad M(Az, Az_1, 2t)M(Az_1, Az, 2t)\, M^2(Az_1, Az_1, t)\} \\ &= \max\{M^3(Az, Az_1, t), 1, 1, M(Az, Az_1, 2t)M(Az_1, Az, 2t)\} \end{aligned}$$

or $M^3(Az, Az_1, kt) \geq 1 \Rightarrow Az = Az_1$.

Thus Az is a unique common fixed point of A, B, S and T.

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ON EVALUATION OF CERTAIN INTEGRALS INVOLVING H-FUNCTIONS

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ABSTRACT. The aim of this paper is to evaluate certain integrals involving H - Functions of one variable.

Key words. H Function, G - Function, Laplace Transformation.

1. INTRODUCTION

Verma, [5] evaluated certain integrals involving G-functions which include certain known results due to Erdelyi [1], Ragab[3] and Saxena [4]. In this paper an attempt has been made to evaluate certain integrals involving H-functions of one variable.

These results are of general nature and include a number of known results as special cases-

2. NOTATIONS AND DEFINITIONS

The H-function of one variable is defined as (Fox [2]).

$$H^{m,n}_{p,q}\left[x\left|\begin{matrix}(\alpha_p\,;a_p)\\(\beta_q\,;b_q)\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi i}\int_L\frac{\Gamma\left[1-(\alpha_n)+(a_n)\xi\right].\Gamma\left[(\beta_m)-(b_m)\xi\right]\,x^{\xi}\,d_{\xi}}{\Gamma\left[(\alpha_{n+1,p})-(a_{n+1,p})\xi\right]\quad\left[1-(\beta_{m+1,q})+(b_{m+1,q})\xi\right]}\tag{1}$$

where L is the usual notation for Barne's type contour integral. The following assumptions are made -

(a) $a_i > 0$; $i = 1, 2, \ldots, p$
$b_j > 0$; $j = 1, 2, \ldots, q$

(b) All the poles of integrand are simple.

(c) $O \le m \le q$ and $O \le n \le p$.

(d) The sequence of parameters $[(\beta_m); (b_m)]$ and $[(\alpha_n); (a_n)]$ are such that the none of the poles coincide. The path of integration is indented, if necessary in such a way that poles of $\Gamma\left[(\beta_m) - (b_m)\xi\right]$ lie on the right and those of $\Gamma\left[1 - (\alpha_n) + (a_n)\xi\right]$ lie on the left of the imaginary axis.

(e) The integral (2.1) converges under certain conditions which can be easily obtained by cosidering the behaviour of the integrand one the closed contour.

We consider the integral in the plane taken round the closed contour L consisting of the imaginary axis from -iR to +iR and that part of semi circle $|\xi| = R$ which lies to the right of the imarginary axis, R being so choosen that the circle always passes between the poles of the integrand.

3. MAIN RESULTS

In this section we establish the following results-

(3.1)
$$\int_0^x t^{b-1} H^{\lambda,\mu}_{\nu,\varepsilon}\left[t^a \left| \begin{matrix} (\alpha_\nu ; A_\nu) \\ (\beta_\varepsilon ; B_\varepsilon) \end{matrix} \right.\right]$$

$$\times H^{S,\;.1+R}_{2+R+T,\;u+s}\left[\frac{x}{t} \left| \begin{matrix} (1-C_R ; \delta_R),\ (1;1),\ (d_s ; \mu'_s) \\ (e_T ; \nu'_T),\ (1-f_u ; \theta_u) \end{matrix} \right.\right] dt$$

$$= x^b H^{\lambda+S,\;.1+\mu+S}_{2+\nu+T+\mu+R,\;\varepsilon+U+\lambda+S}\left[x^a \left| \begin{matrix} (1-b;a),(1-C_R - b\delta_R ; a\delta_R),(\alpha_\nu ; A_\nu),(d_s - b\mu'_s ; a\mu'_s) \\ (e_T + b\nu'_T ; a\nu'_T),(\beta_\varepsilon ; B_\varepsilon),(1-f_U + b\theta_U ; a\theta_U) \end{matrix} \right.\right].$$

$$\int_0^x t^{b-1} H^{\lambda,\mu}_{\nu,\varepsilon}\left[t^{-a} \left| \begin{matrix} (\alpha_\nu ; A_\nu) \\ (\beta_\varepsilon ; B_\varepsilon) \end{matrix} \right.\right] \times H^{S,\;.1+R}_{2+T+R,\;s+U}\left[\frac{1}{tx} \left| \begin{matrix} (1-\sigma_R ; \delta_R)(1;1)(f_s ; \theta_s) \\ (e_T ; \nu'_T)(1-d_u ; \mu'_u) \end{matrix} \right.\right]$$

$$= x^{-b} H^{1+\lambda+S,\, \mu+R}_{\mu+\nu+R+T,\, 2+\varepsilon+\lambda+s+U}\left[x^{a} \left| \begin{matrix} \left(1-e_R+\nu'_R\, b : a\nu'_R\right) . \left(\alpha_\nu : A_\nu\right) . \left(1-f_s+b\theta_s : a\theta_S\right) \\ \left(\sigma_T+\delta_T\, b : a\delta_T\right) \left(\beta_\varepsilon : B_\varepsilon\right) \left(f_u-\theta_u\, b : a\theta_u\right) \end{matrix} \right. \right]$$

(3.2)

Proof. To prove (3.1); let us consider,

$$\int_{o}^{\infty} e^{-rt}\, t^{b-1} . H^{\lambda,\mu}_{\nu,\varepsilon}\left[xt^{a} \left| \begin{matrix} \left(\alpha_\nu ; A_\nu\right) \\ \left(\beta_\varepsilon ; B_\varepsilon\right) \end{matrix} \right. \right] dt .$$

On replacing the H-function in this by its equivalent Barne's contour integral, then changing the order of integration and evaluating the inner integral by Laplace transform, we easily have -

$$\frac{1}{r^{b}} . H^{\lambda,\, 1+\mu}_{1+\nu,\varepsilon}\left[\frac{x}{r^{a}} \left| \begin{matrix} \left(1-b ; a\right), \left(\alpha_\nu ; A_\nu\right) \\ \left(\beta_\varepsilon ; B_\varepsilon\right) \end{matrix} \right. \right] .$$

Putting r=1 ; in this result, then we get;

$$\int_{o}^{\infty} e^{-t}\, t^{b-1} . H^{\lambda,\mu}_{\nu,\varepsilon}\left[xt^{a} \left| \begin{matrix} \left(\alpha_\nu ; A_\nu\right) \\ \left(\beta_\varepsilon ; B_\varepsilon\right) \end{matrix} \right. \right] dt .$$

$$= H^{\lambda,\, 1+\mu}_{1+\nu,\varepsilon}\left[x \left| \begin{matrix} \left(1-b ; a\right), \left(\alpha_\nu ; A_\nu\right) \\ \left(\beta_\varepsilon ; B_\varepsilon\right) \end{matrix} \right. \right]$$

$$\Rightarrow \int_{o}^{\infty} e^{b-1} H^{\lambda,\mu}_{\nu,\varepsilon}\left[xt^{a} \left| \begin{matrix} (\alpha_{\nu};A_{\nu}) \\ (\beta_{\varepsilon};B_{\varepsilon}) \end{matrix} \right. \right] \times H^{0,1}_{1,0}\left[\frac{1}{t} \left| (1;1) \right. \right] dt$$

$$= H^{\lambda,1+\mu}_{1+\nu,\varepsilon}\left[x \left| \begin{matrix} (1-b;a)(\alpha_{\nu};A_{\nu}) \\ (\beta_{\varepsilon};B_{\varepsilon}) \end{matrix} \right. \right].$$

Now we replace x by x^{a} and t by t/x in above to have

$$\int_{o}^{\infty} t^{b-1} H^{\lambda,\mu}_{\nu,\varepsilon}\left[t^{a} \left| \begin{matrix} (\alpha_{\nu};A_{\nu}) \\ (\beta_{\varepsilon};B_{\varepsilon}) \end{matrix} \right. \right] \times H^{0,1}_{1,o}\left[\frac{x}{t} \left| (1;1) \right. \right] dt$$

$$= x^{b} H^{\lambda,1+\mu}_{1+\nu,\varepsilon}\left[x^{a} \left| \begin{matrix} (1-b;a)(\alpha_{\nu};A_{\nu}) \\ (\beta_{\varepsilon};B_{\varepsilon}) \end{matrix} \right. \right]$$

Again we replace x by xy^{δ} and multiplying by e^{-y}, y^{c-1} on both side and integrating w.r.t. y from o to ∞, we get after some simplification.

$$\int_{o}^{\infty} t^{b-1} H^{\lambda,\mu}_{\nu,\varepsilon}\left[t^{a} \left| \begin{matrix} (\alpha_{\nu};A_{\nu}) \\ (\beta_{\varepsilon};B_{\varepsilon}) \end{matrix} \right. \right] \times H^{0,1+R}_{1,\;0}\left[\frac{x}{t} \left| (1-C_{R};\delta_{R}),(1;1) \right. \right] dt$$

$$= x^{b} H^{\lambda,1+\mu+R}_{2+\nu+\mu+R,\varepsilon}\left[x^{a} \left| \begin{matrix} (1-b;a)(1-C_{R}-b\delta_{R};a\delta_{R}),(\alpha_{\nu};A\nu) \\ (\beta_{\varepsilon};B_{\varepsilon}) \end{matrix} \right. \right]$$

Again if we replace x by $xy^{\mu'}$ followed by multiplication of $e^{y}\,y^{-d}$ on both sides then integrating w.r.t. y from o to ∞, using Hankel's contour integral then multiplying $e^{-y}\,y^{c-1}$ and then integrating w.r.t. y from o to ∞ and lastly we multiply $e^{-y}.y^{-f}$ and integrating w.r.t. y from o to ∞ and replace x by $x/y^{\nu'}$ and x/y^{θ} respectively; then we get ;

$$\int_{o}^{\infty} t^{\,b-1} H_{\nu,\varepsilon}^{\lambda,\mu}\left[t^{a}\left|\begin{matrix}(\alpha_{\nu};A_{\nu})\\(\beta_{\varepsilon};B_{\varepsilon})\end{matrix}\right.\right]\times H_{2+R+T,\,U+S}^{S,\,1+R}\left[\frac{x}{t}\left|\begin{matrix}(1-C_{R};\delta_{R}),(1;1),(d_{s};\mu'_{s})\\(e_{T};\nu'_{T}),(1-f_{u};\theta_{u})\end{matrix}\right.\right]$$

$$=x^{\,b} H_{2+\nu+T+\mu+R,\,\varepsilon+U+\lambda+S}^{\lambda+S,\,1+\mu+R}\left[x^{a}\left|\begin{matrix}(1-b;a)(1-C_{R}-b\delta_{R};a\delta_{R})(\alpha_{\nu};A_{\nu}),(d_{s}-b\mu'_{s};a\mu'_{s})\\(e_{t}+b\nu'_{T};a\nu'_{T}),(\beta_{\varepsilon};B_{\varepsilon}).(1-f_{u}+b\theta_{u};a\theta_{u})\end{matrix}\right.\right]$$

The convergence condition of (3.1) is as follows.

In the neighbourhood of $t\rightarrow\infty$, the H-function is of the order of some power of t if $\nu<\varepsilon$ and

$$\mu\geq 1,\lambda+\mu>\frac{1}{2}\nu+\frac{1}{2}\varepsilon \quad \text{and}$$

$$\left|\arg t\right|<\left(\lambda+\mu-\frac{1}{2}\nu-\frac{1}{2}\varepsilon\right)\pi .$$

Further, in order to establish (3.2), we consider

$$\int_{o}^{\infty} e^{-tx}\, t^{\,b-1} H_{\nu,\varepsilon}^{\lambda,\mu}\left[t^{-a}\left|\begin{matrix}(\alpha_{\nu};A_{\nu})\\(\beta_{\varepsilon};B_{\varepsilon})\end{matrix}\right.\right]dt .$$

By Laplace transform, we get,

$$\int_{o}^{\infty} t^{\,b-1} H_{\nu,\varepsilon}^{\lambda,\mu}\left[t^{-a}\left|\begin{matrix}(\alpha_{\nu};A_{\nu})\\(\beta_{\varepsilon};B_{\varepsilon})\end{matrix}\right.\right]\times H_{1,0}^{0,1}\left[\frac{1}{tx}\middle|(1;1)\right]dt .$$

$$= x^{-b} H^{1+\lambda,\mu}_{\nu,1+\varepsilon}\left[x^{a}\left|\begin{matrix}(\alpha_\nu;A_\nu)\\(b;a)(\beta_\varepsilon;B_\varepsilon)\end{matrix}\right.\right]$$

If we replace x by $\frac{x}{y^\delta}, \frac{x}{y^\theta}, \frac{x}{y^\mu}$ and $\frac{x}{y^\nu}$ and mlultiplying $e^{-y}y^{\sigma-1}$, $e^{y}.y^{-f}$, $e^{y}.y^{-d}$ $e^{-y}.y^{e-1}$ integrating w.r.t. y on both sides respectively and using in place of Hankel's transform; Then we get,

$$\int_0^\infty t^{b-1} H^{\lambda,\mu}_{\nu,\varepsilon}\left[t^{-a}\left|\begin{matrix}(\alpha_\nu;A_\nu)\\(\beta_\varepsilon;B_\varepsilon)\end{matrix}\right.\right] H^{S,\ 1+R}_{2+T+R,S+u}\left[\frac{1}{tx}\left|\begin{matrix}(1-\sigma_R;\delta_R)(1;1)(f_S;\theta_S)\\(e_T;\nu'_T)(1-du;\mu'_u)\end{matrix}\right.\right]dt.$$

$$= x^{-b} H^{1+\lambda+S,\mu+R}_{\mu+\nu+R+T,\,2+\varepsilon+\lambda+S+u}\left[x^{a}\left|\begin{matrix}(1-e_R+\nu'_R b:a\nu'_R).(\alpha_\nu:A_\nu).(1-f_S+0_S b:a0_S)\\(b;a)(\sigma_T+\delta_T b:a\delta_T)(\beta_\varepsilon:B_\varepsilon)(f_u-\theta_u b:a\theta_u)\end{matrix}\right.\right]$$

The convergance condition follows as in the previous case.

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ON A RECURRENCE RELATION IN FIBONACCI SEQUENCE

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ABSTRACT. The Fibonacci Sequence is based on an additive relationship between any term and the three preceding terms. We shall make it a linear dependence and it will involve the preceding q terms. $p_n + q = a_1 p_{n+q-1} + a_2 p_{n+q-2} + \dots + a_q p_n$ $(n = 0, 1, 2, 3, \dots)$ where, p - term of polynomial, x- term of power series, P-Fibonacci Number, a_o to a_k - Constant.

Key words. Fibonacci number, R-sequence.

1. INTRODUCTION

The Fibonacci sequence has provoked a remarkable amount of research. It seems that there is no end to the results that may be derived from the basic definition. [10]

$$F_{n+2} = F_{n+1} + F_n \text{ and } F_0 = 0 \text{ and } F_1 = 1 \tag{1.1}$$

Which Leonardo of Pisa found lurking in the simple rabbit problem. An extension of the definition yields the following recurrence relation [11] :

$$P_{n+3} = P_{n+2} + P_{n+1} + P_n \text{ and } P_0 = 0,\ P_1 = 1 \text{ and } P_2 = 1 \tag{1.2}$$

2. LINEAR PROPERTIES OF FIBONACCI SEQUENCE

The Fibonacci sequence is based on an additive relationship between any term and the three preceding terms. We shall make it a linear dependence and it will involve the preceding q terms. Here and throughout, p will note the general additive sequence [6] [9].

$$p_{n+q} = a_1 p_{n+q-1} + a_2 p_{n+q-2} + \dots + a_q p_n \ (n = 0, 1, 2, 3, \dots) \tag{2.1}$$

It seems essential to the spirit of these sequences that they be integral. To ensure this, we must demand that the set $\{a_i\}_1^q$ be integers. This set will be called the spectrum. But, returning to (2.1) and letting n = 0 :

$$p_q = \sum_{k=1}^{q} a_k p_{q-k} \tag{2.2}$$

reveals that we must specify the first q terms of the sequence in order that the other may be obtained. The set of integers $\{p_i\}_0^{q-1}$ so specified will be called the initial set of the initials.

It might be mentioned here that the Fibonacci sequence is obtained by letting q = 3 and taking the spectrum {1,1,2} and the initials {0,1,1}. [2]

We wish now to extend the definition (2.1) so that negative values for the index are allowed. Using the "back-up" approach, we obtain [7]

$$p_{q-1} = a_1 p_{q-2} + a_2 p_{q-3} + \ldots + a_{q-1} p_0 + a_q p_{-1}$$

or

$$p_{-1} = \frac{1}{a_p}\left(p_{q-1} - \sum_{k=1}^{q-1} a_k p_{q-1-k}\right)$$

continuing, it can be seen that for any n = 0, 1, 2, ...

$$p_{-1} = \frac{1}{a_p}\left(p_{q-n} - \sum_{k=1}^{q-1} a_k p_{q-n-k}\right) \tag{2.3}$$

Clearly, in order to maintain an integral sequence for all values of the index, positive and negative, it is necessary to take aq = + 1. In any case, we have that $a_p^2 = 1$.

3. ALGEBRAIC GENERATORS

One of the most common manifestations of additive recursive sequences is the power series expansion of certain functions. For example, a short calculation leads one to conclude that [8] [12] :

$$\frac{x}{1-x-x^2-x^3}=\sum_{k=0}^{\infty} p_k x^k$$

The actual derivation of this result stems directly from the definition

of the Fibonacci sequence. In what follows, we will use the same derivation in a generalized form.

What we want to discover is an expression for :

$$\sum_{k=0}^{\infty} p_k x^k = \phi(x) \tag{3.1}$$

where, by (2.1)

$$p_{n+q} = \sum_{k=1}^{q} a_k p_{n+q-k} \tag{3.2}$$

Now, we multiply (2.1) by x^{n+q}, and sum over the index n, so that :

$$\sum_{n=0}^{\infty} p_{n+q} x^{n+q} = \sum_{k=1}^{q} a_k x^k \sum_{n=0}^{\infty} p_{n+q-k} x^{n+q-k} \tag{3.3}$$

But taking into (3.1) we may rearrange this expression, and :

$$\phi(x)(1-\sum_{k=1}^{q} a_k x^k = p_o + \sum_{k=1}^{q-1} x^k \left(p_k \sum_{j=1}^{k} a_j p_{k-j} \right) \tag{3.4}$$

This singularly awkward expression can be made manageable by making the some what arbitrary definition of a0 = -1. The introduction of a0 greatly simplifies the formulation of the required function [14] :

$$\phi(x) = \frac{\sum_{k=0}^{q-1} x^k \sum_{j=0}^{k} a_j p_{k-j}}{\sum_{k=0}^{q} a_k x^k} \tag{3.5}$$

We need hardly say that this is the required expression, which reduces to the familiar Fibonacci power series when $q=3$, $a1=a2=1$, $a3=2$ and $p0 = 0$, $p1 = 1$ and $p2 = 1$. But further investigation of (3.5) leads to considerations which will be of crucial importance later. First, we remark that the denominator is a q-th degree polynomial [4] :

$$- a_0 + a_1 x + a_2 x^2 + \dots + a_q x^q, \ (a_0 = -1)$$

which will be called the spectral polynomial.

Then, with regard to the numerator, the following definition will be made :

$$g_{m,k} = \sum_{q=0}^{k} a_j p_{m+k-j}, \text{ for } 0 < k < q \tag{3.6}$$

In other words, $g_{m,k}$ is a partial sum of terms, for example :

$$g_{m,0} = - a_0 p_m = p_m$$
$$g_{m,q-1} = a_q p_{m-1}$$

and, for convenience :

$$g_k = g_{0,k} = p_k - a_1 p_{k-1} - \dots - a_k p_0 \tag{3.7}$$

The introduction of (3.7) into (3.5) yields the remarkably concise :

$$\phi(x) = \sum_{k=0}^{\infty} p_k x^k = \frac{-\sum_{k=0}^{q-1} g_k x^k}{\sum_{k=0}^{q} a_k x^k} \tag{3.8}$$

4. THE R-SEQUENCE

In any p-sequence, it is possible to choose the initial set as any set of q consecutive terms. It seems then necessary to consider some sort of fundamental sequence. This fundamental sequence has the simplest non-trivial initial set, namely $\{0, 0, 0, \ldots, 0, 1, 1\}$. The Fibonacci sequence is a tertiary case. These sequence exist for all values of q, and they will be called R-sequences. Referring to (3.6), we will rename the partial sums $g_{m,k}$ [13] :

$$G_{m,k} = -\sum_{j=0}^{k} a_j R_{m+k-j}, \; 0 < k < q \qquad (4.1)$$

And from (3.7)

$$G_k = G_{0,k} = -\sum_{j=0}^{k} a_j R_{k-j}$$

But from the definition of R-sequence, $R_k = 0$ for $0 < k < q-3$ and $R_{q-1} = 1$

$$G_k = 0, \text{ for } 0 < k < q-2 \text{ or } k = q$$
$$G_{q-1} = 1$$

Using these results in (3.8), we have :

$$\sum_{k=0}^{\infty} R_k x^k = \frac{-\sum_{k=0}^{q-1} G_k x^k}{\sum_{k=0}^{q} a_k x^k} = \frac{x^{q-1}}{-\sum_{k=0}^{q} a_k x^k} \qquad (4.2)$$

The right-hand member of (4.2) may be treated as a geometric series :

$$\frac{x^{q-1}}{1-\sum_{k=1}^{q} a_k x^k} = x^{q-1}\left[\sum_{k_1=0}^{\infty} \sum_{k=0}^{q} \left(a_k x^k\right)^{k_1}\right]$$

and successive binomial expansions of the polynomial in parenthesis gives

$$= x^{q-1}\left[\sum_{k_1=0}^{\infty}\sum_{k=0}^{q}\left(a_k x^k\right)^{k_1}\right]$$

$$= x^{q-1}\left[\sum_{k_1=0}^{\infty}\sum_{k=0}^{q}\binom{k_1}{k_2}(a_1x)^{k_1-k_2}\sum_{k=2}^{q}\left(a_k x^k\right)^{k_2}\right]$$

and so on. After q steps, we may collect coefficients of $xm+q-1$ in (4.1) and equate them, obtaining :

$$R_{m+q-1} = \sum \binom{k_1}{k_2}\binom{k_2}{k_3}\binom{k_3}{k_4}\binom{k_{q-1}}{k_q} a_1^{\,k_1-k_2} a_2^{\,k_2-k_3} \ldots a_{q-1}^{\,k_{q-1}-k_q} a_q^{\,k_{q1}} \qquad (4.3)$$

Where the sum is taken over all $\{k_i\}$ such that : $\sum_{i=1}^{q} k_i = m$

and $m+q-1 \geq 0$. Looking at the binary case ($q = 2$), we get that

$$R_{n+1} = \sum_{i=0}^{\infty}\binom{n-i}{i} a_1^{\,n-2i} a_2^{\,i} \qquad (4.4)$$

Where $\binom{n}{k} = 0$ for $0 < n < k$ and $n+1 > 0$, so that for $a1 = a2 = 1$:

$$P_{n+1} = \sum_{i=0}^{\infty}(n_i - i)$$

Which, of course, is the well-known "rising diagonal" result for Fibonacci numbers, and here is the ternary case ($q = 3$) :

$$R_{n+2} = \sum_{i,j}\binom{n-i-j}{i}\binom{i}{j} a_1^{\,n-2i-j} a_2^{\,i-j} a_3^{\,j} \qquad (4.5)$$

We choose to make agreement that $\binom{n}{k} = 0$ for all $0 < n < k$. This appears a bit arbitrary, but it is used since it simplifies the summation notion.

Notice that the upper index on the summation many be taken as infinity, since by our agreement the binomial coefficient vanish for large enough k [1].

5. THE R-SEQUENCE AS BASIS

The fundamental nature of the R-sequence is clearly shown in the following argument. We return to (3.8) and rearrange slightly [5] :

$$px^k = \frac{-\sum_{k=0}^{q-1} g_k x^k}{\sum_{k=0}^{q} a_k x^k} = \sum_{k=0}^{q-1} g_k x^{k-q+1}\left(\frac{x^{q-1}}{\sum_{k=0}^{q} a_k x^k}\right)$$

then taking into account (4.2), we have :

$$\sum_{k=0}^{\infty} p_k x^k = \sum_{k=0}^{q-1} g_k x^{k-q+1}\left(\sum_{i=0}^{\infty} R_i x^i\right)$$

Comparing the coefficients of x^{m-q+1} in this expression, we find that :

$$P_{m-q+1} = \sum_{k=0}^{q-1} g_k R_{m-k} \tag{5.1}$$

This clearly shows the basic nature of the R-sequence, it forms a basis set for any other sequence having the same spectrum. But a more useful formulation may be derived by considering (3.7), and substituting into (5.1) [8] [3].

$$P_{m-q+1} = -\sum_{k=0}^{q-1} \sum_{j=0}^{k} a_j p_{k-j} R_{m-k} \tag{5.2}$$

And then by using (4.1) : $-\sum_{j=0}^{k} a_j R_{m-q+1+k-j} = G_{m-q+1,k}$

we have from (5.2) :

$$P_{m-q+1} = \sum_{k=0}^{q-1} G_{m-k+1,k} P_{q-k-1} \qquad (5.3)$$

And finally, on obvious adjustment of index leaves us with :

$$P_m = \sum_{k=0}^{q-1} G_{m,k} p_{q-k-1}, \qquad (5.4)$$

where $G_{m,k}$ is given by (5.3).

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